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SOLVING DIFFERENCE EQUATIONS BY FORWARD DIFFERENCE OPERATOR METHOD

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ABSTRACT

In this paper a forward difference operator method was used to solve a set of difference equations. We also find the particular solution of the nonhomogeneous difference equations with constant coefficients. In this case, a new operator call the forward difference operator $\Delta r,s$, defined as $\Delta_{r,s}\, y_n = r\, y_{n+1} - s\, y_n$, was introduced. Some of the properties of this new operator were also investigated.

Keywords: forward difference, particular solution, nonhomogeneous, difference equations, constant coefficients.

INTRODUCTION

In Numerical Analysis, we use some linear operators such as shift exponential operator E, with $Efj = fj+1$, forward difference operator Δ, with $\Delta fj = fj+1 - fj$, and backward difference ∇, with $\nabla fj = fj - fj-1$. These operators are used in some aspects of Numerical Analysis, particularly in interpolation, quadratures, difference equations, and so forth. (Odior, 2003; Lambert, 1973; Phillips *et al*. 1980).

Under the forward difference operator Δ, the linear difference equations are written in one of the following forms

$$P(\Delta)y_n = 0, \text{ (homogeneous)} \qquad (1)$$

$$P(\Delta)y_n = f_n \text{ (nonhomogeneous)} \qquad (2)$$

Where *P* is a polynomial.

Prototype results towards the development of the basic theory of the global behaviour of solutions of nonlinear difference equations of order greater than one. The techniques and results are also extremely useful in analyzing the equations in the mathematical models of various biological systems and other applications (Kalabu and Kulenovi, 2003; senior, *et al*. 2001).

THEOREM

To solve linear difference equations for general solution, we introduce a set of theorems which are used for the general solution, (Hosseinzadeh and Afrouzi, 2007).

Theorem 1

This is the principle of superposition. Suppose that $y_1, y_2, ..., y_m$ are the solutions of the homogeneous difference Equation (1), then any linear combinations of them is a solution for same equation too.

Theorem 2

Suppose that the complex-valued function, $y_n = y_1 + iy_2$ be a solution of equation (1), then functions y_1, y_2 are also solutions for the equation.

Theorem 3

Let y_h be a solution for (1) and y_p be a particular solution for (2), then $y_c = y_h + y_p$ is a solution for (2) too.

Finite differences

Given a special function of f(x), with an argument x proceeding at equal intervals, we could generate a Table of logarithms or trigonometric functions or a Table of special computation for certain calculation (Odior, 2003). Given a certain function of x, say, with values f (a), f (a+b), f (a+2h),….., the difference between the consecutive values of x, denoted by h is called the interval differencing.

If $f(x_1) = f(x)$
$f(x_2) = f(x + h)$ and
$f(x_2) - f(x_1) = \Delta f(x)$
then $\Delta f(x) = f(x + h) - f(x)$ (3)

Where Δ is called "difference operator" and $\Delta f(x)$ is called the first difference of f(x). The second difference of f(x) is given thus:

$$\Delta[\Delta f(x)] = \Delta^2 f(x) = \Delta f(x+h) - \Delta f(x) \qquad (4)$$

Building a difference Table

Build up a difference Table for the function, $f(x) = x^3 + 3x^2 - 2x + 5$, taking x = 0 as the initial value, and h = 2.

Solution: We now build up the Table as follows:

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x	f(x)	Δf(x)	Δ²f(x)	Δ³f(x)	Δ⁴f(x)
0	5				
		16			
			72		
2	21				
		88		48	
			120		0
4	109				
		208		48	
			168		0
6	317				
		376		48	
			216		
8	693				
		592			
10	1285				

The Operators E and Δ
The Newton-Gregory forward difference formula for f_p is given as:

$$f_p = E^p f_0 = f_0 + \binom{p}{I}\Delta f_0 + \binom{p}{2}\Delta^2 f_0 + \binom{p}{3}\Delta^3 f_0 + \ldots..$$

This is obtained from,

$$f_{x+nh} = E^n f_x = (1+\Delta)^n f_x$$

$$= f_x + \binom{n}{1}\Delta f_x + \binom{n}{2}\Delta^2 f_x + \binom{n}{3}\Delta^3 f_x + \ldots.. \quad (5)$$

Direction of errors in a Table of differences

The effect of a single error in a Table of differences from the point of error becomes considerably magnified. This is illustrated in the difference Table bellow, in which error is made in the value f_4.

f_1						
	Δf_1					
		$\Delta^2 f_1$				
f_2						
	Δf_2		$\Delta^3 f_1 + \varepsilon$			
				$\Delta^4 f_1 - 4\varepsilon$		
f_3		$\Delta^2 f_2 + \varepsilon$				
	$\Delta f_3 + \varepsilon$		$\Delta^3 f_2 - 3\varepsilon$		$\Delta^5 f_1 - 20\varepsilon$	
f_4		$\Delta^2 f_3 - 2\varepsilon$		$\Delta^4 f_2 + 6\varepsilon$		$\Delta^6 f_1 - 20\varepsilon$
	$\Delta f_4 - \varepsilon$		$\Delta^3 f_3 + 3\varepsilon$		$\Delta^5 f_2 - 10\varepsilon$	
f_5		$\Delta^2 f_4 + \varepsilon$		$\Delta^4 f_3 - 4\varepsilon$		
	Δf_5		$\Delta^3 f_4 - \varepsilon$			
f_6		$\Delta^2 f_5$				
	Δf_6					
f_7						

So, it is clear that the error made in the value f4 will continue to affect some of the other values generated until it is pronounced in all the values.

Difference equation

A difference equation is an equation involving the differences between successive values of a function of an integer variable. It can be regarded as the discrete version of a differential equation. For example the difference equation $f(n+1) - f(n) = g(n)$ is the discrete version of the differential equation $f'(x) = g(x)$. We can see difference equation from at least three points of views: as sequence of number, discrete dynamical system and iterated function. It is the same thing but we look at different angle. Difference equation is a sequence of numbers that are generated recursively using a rule to relate each number in the sequence to previous numbers in the sequence as presented in Figure-1, {1, 1, 2, 3, 5, 8, 13, 21,…}. The Sequence {1, 1, 2, 3, 5, 8, 13, 21,….} is called Fibonacci sequence, generated with rule $y(k+2) = y(k+1) + y(k)$ for $k = 0, 1, 2, 3,\ldots$, and initial value $y(0) = y_0 = 1$.

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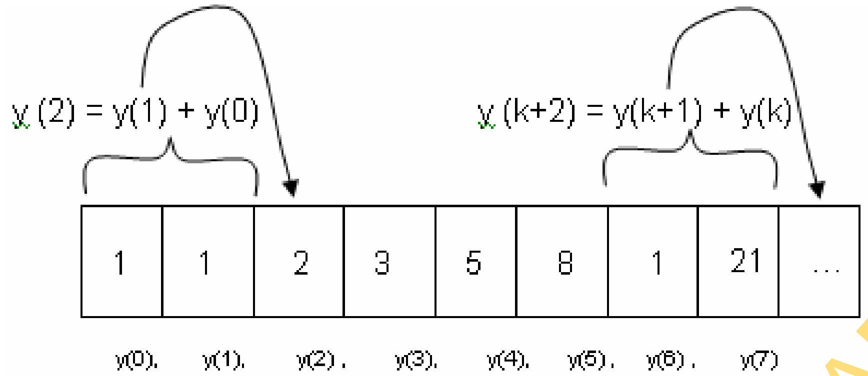

Figure-1. Sequence of difference equation.

Sequence {3, 5, 7, 9, ….} has rule y(k+1) = 2y(k) + 3 for k = 0, 1, 2, 3,…,.

Both sequences have initial value y (0) = y_0 = 0.

MATERIALS AND METHODS

We now consider the homogeneous and nonhomogeneous difference equations with constant coefficients with application of the approach considered by Hassan and Afrouzi1 (2008).

Let $y_n = r^n$ be a solution for equation (1), we have

$$P(r - 1) = 0. \qquad (6)$$

Where Equation (3) is the corresponding characteristic equation to Equation (1).

Remarks:

The following remarks can be drawn from the forgoing as presented below:

Remark 1: All roots of the characteristic equations may be distinct real values, either some of them equal or some of them are conjugate complex number.

(i) If r_1, r_2, …., r_k be the distinct real roots to the characteristic equations, then the functions r_1^n, r_2^n,….,

r_k^n will be solutions of the homogeneous equations.

Then the above functions are the fundamental solutions of the homogeneous equation.

(ii) If $r_1 = r_2 = ... = r_m = r$ be the repeated roots of the characteristic equation (3), then the fundamental solutions of the homogeneous equation are: r^n, nr^n, n^2r^n, …., $n^{m-1}r^n$ and they are linearly independent, (Lambert, 1973).

(iii) If $r_{1,2} = \alpha \pm i\beta$ be two conjugate complex roots, the fundamental solutions of the homogeneous equation are,

$$y_1 = (\alpha^2 + \beta^2)^{n/2} \cos n\varphi,\ y_2 = (\alpha^2 + \beta^2)^{n/2} \sin n\varphi,$$

Where $\varphi = \tan^{-1} (\beta/\alpha)$.

Fundamental solution

(1): Find the fundamental solutions of the following homogeneous equation

$(\Delta^4 + \Delta^2)y_n = 0.$

Solution: The characteristic equation is $(r - 1)^2(r^2 - 2r + 2) = 0$. This polynomial equation has one double root; ($r = 1$) and two complex conjugate roots ($r = 1 \pm i$), therefore the fundamental solutions become:

$$y_1 = 1,\ y_2 = n,\ y_3 = 2^{\frac{n}{2}} \cos\frac{n\pi}{4} \text{ and } y_4 = 2^{\frac{n}{2}} \sin\frac{n\pi}{4}$$

(2): Solve the following difference equation and find the fundamental solutions

$(5\Delta + 6)(32\Delta^2 + 56\Delta + 25)y_n = 0$

Solution: The roots of the corresponding characteristic equation are $r_1 = \frac{-1}{5}$,

$r_{2,3} = \frac{1}{8}(1 \pm i)$. This gives the solutions of the equation to be:

$$y_1 = (\frac{-1}{5})^n,\ y_2 = 2^{-\frac{5n}{2}} \cos\frac{n\pi}{4} \text{ and } y_3 = 2^{-\frac{5n}{2}} \sin\frac{n\pi}{4}$$

(3): Find the fundamental solutions of the following homogeneous equation

$(\Delta^4 + \Delta^2)y_n = 0.$

Solution: The characteristic equation is $(r - 1)^2 (r^2 - 2r + 2) = 0$. This polynomial equation has one real double root $r = 1$ and two complex conjugate roots $r = 1 \pm i$, therefore the fundamental solutions may be written as follow:

$$y_1 = 1,\quad y_2 = n,\quad y_3 = 2^{\frac{n}{2}} \cos\frac{n\pi}{4} \text{ and } y_4 = 2^{\frac{n}{2}} \sin\frac{n\pi}{4}.$$

We need to prove the accuracy of the following equalities in our operations.

(i) $\Delta \sum_{j=0}^{n-1} f_j = f_n$

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(ii) $\frac{1}{\Delta} f_n = \sum_{j=0}^{n-1} f_j$

For the proof of $\Delta \sum_{j=0}^{n-1} f_j = f_n$, we consider

$$\Delta \sum_{j=0}^{n-1} f_j = \sum_{j=0}^{n} f_j - \sum_{j=0}^{n-1} f_j = f_n.$$

$$\therefore \ \Delta \sum_{j=0}^{n-1} f_j = f_n.$$

For the proof of $\frac{1}{\Delta} f_n = \sum_{j=0}^{n-1} f_j$, we consider

$$\Delta \sum_{j=0}^{n-1} f_j = \sum_{j=0}^{n} f_j - \sum_{j=0}^{n-1} f_j = f_n,$$

and $\Delta \sum_{j=0}^{n-1} f_j = f_n$

$$\therefore \ \frac{1}{\Delta} f_n = \sum_{j=0}^{n-1} f_j.$$

Each of the above identities is used for finding particular solution of non-homogeneous difference equations with constant coefficients.

(4): Find the particular solution of the following difference equation

$\Delta y_n = \cos n\beta$

Solution:

$\Delta y_n = \cos n\beta$ and so $y_n = \frac{1}{\Delta}(\cos n\beta)$

Let $y_p = \frac{1}{\Delta}(\cos n\beta) = \sum_{j=0}^{n-1} \cos j\beta = \frac{1}{2}\sin(\frac{2n-1}{2})\beta \cos\frac{\beta}{2} - \frac{1}{2}$

(5): Find the particular solution of the following difference equation

$$\Delta^2 x_m = \cos(\frac{m+1}{3})\pi$$

Solution:

The solution is obtained thus:

$$x_m = \frac{1}{\Delta}\cos(\frac{m+1}{3})\pi = \frac{1}{\Delta}\sum_{j=0}^{m-1}\cos\frac{(j+1)\pi}{3} = \sum_{j=0}^{m-1}(-\frac{1}{2} + \sin\frac{(2j+1)\pi}{6}) = 1 - \frac{1}{2}m - \cos\frac{m\pi}{3}$$

RESULTS AND DISCUSSIONS

We define the forward difference operator $\Delta_{r,s}$ as follow

$\Delta_{r,s} y_n = r y_{n+1} - s y_n = (rE - s) y_n$

Where y_n is the approximate value function $y(x)$ at point $x_n \in [x_0, x_m]$, then two operators $(\Delta r,s)$ and $(rE - s)$ are equivalent.

Difference Operations in Vector Space of Operator $\Delta r,s$. Some basic difference operations in the vector space of difference operator are defined as:

(i) $\Delta r_1,s + \Delta r_2,s \equiv \Delta r_1 + r_2,s$
(ii) $\Delta r,s_1 + \Delta r,s_2 \equiv \Delta r,s_1 + s_2$
(iii) $\Delta r_1,s - \Delta r_2,s \equiv \Delta r_1 - r_2,s$
(iv) $\Delta r,s_1 - \Delta r,s_2 \equiv \Delta r,s_1 - s_2$
(v) $\Delta r_1,s_1 \times \Delta r_2,s_2 \equiv \Delta r_2,s_2 \times \Delta r_1,$
(vi) $\frac{\Delta_{r1,s1}}{\Delta_{r2,s2}} \equiv \Delta_{r1,s1}(\frac{1}{\Delta_{r2,s2}}) \equiv (\frac{1}{\Delta_{r2,s2}})\Delta_{r1,s1}$

Solution of nonhomogeneous difference equation with constant coefficients

A general form of nonhomogeneous difference equation with order m can be written in the form

$$P(E)y_n = f_n, \quad P(\Delta)y_n = f_n, \quad P(\nabla) = f_n \qquad (7)$$

This can as well be written as:

$$y_p = \frac{1}{\prod_{j=1}^{m}\Delta_{rj,sj}} f_n = \frac{1}{\Delta_{rm,sm}}(\frac{1}{\Delta_{rm-1,sm-1}}(\text{L} \ \frac{1}{\Delta_{r1,s1}} f_n \text{L}))s$$

To find the particular solution of a given nonhomogeneous difference equation such as:

$\Delta_{2,1}\Delta_{2,3} y_n = 2^n n^2,$

We have: Let $y_p = \frac{1}{\Delta_{2,1}\Delta_{2,3}}(2^2 n^n)$

$$= \frac{1}{\Delta_{2,1}}(\frac{1}{\Delta_{2,3}}(2^n n^2)) = \frac{1}{\Delta_{2,1}}(2^n(n^2 - 8n + 28)) = 2^n(\frac{1}{3}n^2 - \frac{32}{9}n + \frac{368}{27})$$

CONCLUSIONS

We have been able to establish the first difference, second difference, third difference, and forth difference of a given polynomial function. The shift operator E method for solving the non-homogeneous difference equations with constant coefficients is a new method which can be used to solve all types of nonhomogeneous difference equations with constant coefficients.

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