DEVELOPMENT OF ALTERNATIVE LINEAR ESTIMATORS IN COMPLEX SURVEYS

by

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ABSTRACT

The estimation of multiple characteristics using Probability Proportional to Size (PPS) sampling scheme has introduced some complexities in sample surveys. It requires transformation of auxiliary information into probability measures and the utilization of correlation coefficient between study variables y and measure of size x. Existing estimators of finite population characteristics are rigidly specified by a fixed order of positive correlation between y and x and are assumed efficient for all populations. However, the assumptions break down when the study variables are negatively correlated with measure of size. In this study, a linear class of estimators that are functions of moments in positive and negative correlation coefficients were proposed.

Using laws of proportions and probability measure theory, a class of alternative linear estimators $\hat{\tau}_{g,c}$ were developed for use in PPS sampling schemes. Using linear regression model with slope β and well-behaved error term ε, the expectation of c^{th} standardized moment of the study variable given by $E\left(\frac{y-\mu_y}{\sigma_y}\right)^c = E\left[\beta\left(\frac{x-\mu_x}{\sigma_y}\right)+\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_y}\right)\right]^c, c = 1,2,3,4$ with $\beta^c = \left(\rho^2\frac{\sigma_y^2}{\sigma_x^2}\right)^{\frac{c}{2}}$ provided a link between moments in correlation coefficient and distribution of the target population, where ρ is the correlation coefficient, $\mu_y, \mu_x, \mu_\varepsilon$ and $\sigma_{y,}^2 \sigma_x^2, \sigma_\varepsilon^2$ are means and variances of $y, x\ \varepsilon$ respectively. The minimum variance was used as optimality criterion for comparing the performance of $\hat{\tau}_{g,c}$ with the conventional estimator namely, Hansen and Hurwitz's estimator $\hat{\tau}_{HH}$, and other existing alternative estimators namely, Amahia-Chaubey-Rao's estimator ($\hat{\tau}_{ACR}$), Grewal's estimator ($\hat{\tau}_G$), Rao's estimator ($\hat{\tau}_R$) and Ekaette's estimator ($\hat{\tau}_E$) under the PPS sampling design. Using the general super-population model with parameter g, the expected Mean Square Error (ξMSE) was derived for the estimators and their relative efficiencies were then computed. Empirical studies with samples drawn from four populations, namely; Population I, II, III and IV having correlation coefficients, $\rho = 0.16, 0.39, -0.32 \text{ and } -0.775$ respectively were conducted.

The derived transformation for generalized selection probabilities defining the class of linear estimators is $p_{i,g}^* = \frac{1-\rho^c}{N} + \rho^c \wp_i;\ c = 1,2,3,4$ where $\wp_i = \frac{x_i}{X}, X = \sum_i^N x_i$ or $\wp_i = \frac{z_i}{Z}, Z = \sum_i^N z_i, z_i = \frac{1}{x_i}$ for positive and negative correlations respectively. Provided that $CV_x < CV_y, \gamma_y < \gamma_x, K_y < K_x$ and $\rho^2 < 1$ for both positive and negative correlations where CV_y, γ_y, K_y and CV_x, γ_x, K_x are coefficients of variation, skewness and kurtosis of x and y respectively and ρ^2 is the coefficient of determination, $\hat{\tau}_{g,c}$ with $c = 2$ was the best estimator for population II, while $\hat{\tau}_{g,C}$ with $c = 1$ was the best estimator for population I in terms of relative mean square error for positive correlation. Under the same conditions and for negative correlation, $\hat{\tau}_{g,c}$ with $c = 2$ and 4 were the best estimators for populations III and IV respectively in terms of relative mean square error. At $g = 0$, $\xi MSE(\hat{\tau}_1) = 131.293 < \xi MSE(\hat{\tau}_{HH}) = 134.3, \xi MSE(\hat{\tau}_2) = 826.5 < \xi MSE(\hat{\tau}_{HH}) = 1043.0, \xi MSE(\hat{\tau}_2) = 254.3 < \xi MSE(\hat{\tau}_{HH}) = 329.7$ and $\xi MSE(\hat{\tau}_4) = 266.3 < \xi MSE(\hat{\tau}_{HH}) = 229.2$ for Population I, II, III and IV respectively. Similarly, when $g = 1$, $\xi MSE(\hat{\tau}_{g,c}) < \xi MSE(\hat{\tau}_{HH})$ for all populations. However, at $g = 2$, $\hat{\tau}_{HH}$ is relatively more efficient than the alternative estimators. All estimators converge to $\hat{\tau}_{HH}$ when $\rho = \pm 1$ and to $\hat{\tau}_R$ when $\rho = 0$.

The developed alternative estimators accommodated all dimensions of correlation coefficients. The derived estimators also reflected the structure of population distribution and enhanced its power of estimation.

Keywords: Probability proportional to size, Multiple characteristics, Standardized moment, Population distribution.

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DEDICATION

This work is dedicated to God Almighty, to my parents, Mr and Mrs Ikughur Udende, my wife, Helen Bosede and children, ‘avese and ‘kator.

CERTIFICATION

I certify that this work was carried out by Mr. Jonathan Atsua Ikughur in the Department of Statistics, University of Ibadan, Nigeria under my supervision.

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INDEX OF ABBREVIATIONS

SRS	Simple Random Sampling
UPS	Unequal Probability Sampling
PPS	Probability Proportional to Size
PPSWR	Probability Proportional to Size with replacement
PPSWOR	Probability Proportional to Size without Replacement
HLE	Homogenous Linear Estimator
SD(p)	Sampling Design
UMVUE	Uniformly Minimum Variance Unbiased Estimator
HTE	Horvitz and Thompson Estimator
HHE	Hansen and Hurwitz Estimator
UNN `	Uniformly Non Negative
RHCE	Rao-Hartley and Cochran Estimator
BSE	Bansal and Singh's Estimator
ACRE	Amahia, Chaubey and Rao's Estimator
GE	Grewal's Estimator
EE	Ekaette's Estimator
YSGE	Yate, Sen and Grundy's Estimator
SE	Sahoo's Estimator
MHHE	Modified Hansen and Hurwitz Estimator

INDEX OF PRINCIPAL NOTATION

p_i	The i^{th} selection probability
p_i*	The i^{th} generalized selection probability
$\hat{\tau}_c$	Conventional Estimator of population total in PPS sampling
$\hat{\tau}_{g,c}$	The generalized Hansen and Hurwitz Estimator
$MSE(\hat{\tau}_{g,c})$	Design based Mean Squared Error of the generalized Hansen and Hurwitz Estimator
ξMSE	Model Based Means Squared Error (Expected Mean Squared Error)
$\xi V \ (\hat{\tau}_{g,c})$	Model Based Variance of the generalized alternative linear estimator
$V_p(\hat{\tau}_{g,RHC})$	Design Based variance of the generalized Hansen Hurwitz Estimator.
$V_p(\hat{\tau}_{g,c})$	Design Based variance of the generalized alternative linear estimators
$V(\hat{\tau}_{HTE})$	Design Based variance of the Horvitz and Thompson Estimators
$B_p(\hat{\tau}_{g,c})$	Design based Bias of the linear alternative estimators
RE$(\hat{\tau}_{g,c})$	Design Based Relative efficiency of the generalized alternative linear estimators
$\xi RE(\hat{\tau}_{g,c})$	Model Based Relative efficiency of the generalized alternative linear estimators
$CD(x)$	Coefficient of determination of the auxiliary variable x
$CV(x)$	Coefficient of variation of the auxiliary variable x
γ_x	Coefficient of skewness of the auxiliary variable x
K_x	Coefficient of kurtosis of the auxiliary variable x
πPS	Inclusion Probability Proportional to size (pi-PS sampling)
$\rho_{x,y}$	Population correlation coefficient between x and y
Rho	Correlation coefficient.

CHAPTER ONE

INTRODUCTION

1.1 Preliminaries

In this thesis, we consider the development of alternative estimators of population total Y, of a real variable y defined on a survey population of known number N of identifiable units, i = 1,2, ...,N. To realize this, we consider sampling schemes that are considered complex in the sense of differing from simple random sampling (SRS) with replacement (WR) or without replacement (WOR).

We shall consider $\hat{\tau}$, the estimator of population total using both design and model postulations connecting the study variables y and the selection probabilities $p_{i.}$ Throughout this study, finite survey population will be assumed.

For an uninterrupted flow of discussions in later chapters, we present the basic concepts and definitions in this section.

A *finite population* $\wp$ is a collection of a known number, N of identifiable units labelled 1, 2, ..., i, ..., N; $\wp$={1,2,...,i,...,N}, where i stands for the physical unit labelled i. Let the unit y_i be associated with the variable i, (i=1,2,...,N), then we associate vector of real numbers y = (y_1, y_2, ..., y_N) with $\wp$, where y is the study variable which is assumed unknown. Thus, we are interested in estimating a parameter function θ, say, population total,

$$\tau = \sum_{i=1}^{N} y_i \qquad ...1.1$$

and

$$S^2 = (N-1)^{-1} \sum_{i=1}^{N} (y_i - \bar{Y})^2 \qquad ...1.2$$

By choosing a part of the population from $\wp$ and observing the values of y only on the units in the sample. A *sample* is a part of a population. It is drawn either with replacement (WR) or without replacement (WOR). In WR sampling, it is a sequence:

$$s = \{i_1, i_2, \dots, i_n\}; 1 \le i_t \le N, \;\; t = 1,2,\dots,n$$

i_t denoting the result of the t^{th} draw. Here, i_t is not necessary equal to $i_{t'}$. Usually, a non-zero probability of selection is allocated to every unit that is selected in the sample at any specific draw.

Under WOR sampling, a sample is a sequence

$$s = \{i_1, i_2, \dots, i_n\}; 1 \le i_t \le N, i_t \ne i_{t'} \text{ for } t \ne t' (= 1,2,\dots,N),$$

as repetition of units in s is not permitted.

Let **A** be the minimum σ-field over $\wp$ and p be the probability measure defined over **A** such that *p(s)* is the probability of selecting s satisfying

$$\left.\begin{matrix} p(s)>0 \\ \sum_{i=1}^{N} p(s)=1 \end{matrix}\right\} \qquad ...1.3$$

then a random sample is selected using a sampling design **p.** Obviously, a sampling design (SD) is a function defined as

$$P: S \to [0,1]\, ;\, \sum_{s \epsilon S} p(s) = 1 \qquad ...1.4$$

where,

$$S = \{s : s \epsilon \wp\}$$

An ordered design p(s) is called fixed size design (FS design) if n(s) is constant for all $s\epsilon\wp$ such that *p(s)* > 0. For this constant sample size, we have FES(n) design. Godambe(1982) stated that this occurs when

$$[v(s) \ne n] \to [p(s) = 0], s \epsilon S. \qquad ...1.5$$

The probability of inclusion of a population unit in the sample is defined as the total probability assigned to a population unit for being included in the sample in all draws.

Let $s \ni i$ denotes all samples S that include the i unit for a given sampling design and $s \ni i, j$ denote all samples S that include the i^{th} and j^{th}, $j \ne i$, $1 \le j \le N$, then the first order and second order inclusion probabilities are π_i and π_{ij} respectively, defined by

$$\pi_i = \sum_{s \ni i} p(s) \qquad ...1.6$$

and

$$\pi_{i,j} = \sum_{s \ni i,j} p(s) \qquad ...1.7$$

A sampling scheme (SS) gives the conditional probability of drawing a unit at any draw given the result of the previous draws. It specifies the conditional probabilities

$$p_r(i_r \backslash i_{1,}, \dots, i_{r-1}) \qquad ...1.8$$

Hanurav(1962) stated that for any given sample design, there exist at least one sampling scheme which realizes this design.

Now, when a sample has been selected, data is usually collected from the field. We define this data set as:

$$d = \{(k, y_k), k \in S\} \qquad ...1.9$$

It is assumed here that the data so collected is free from response and measurement errors and is correct. Upon this data set and using the PPS sampling scheme, an estimator

$$\hat{\tau} = e(s, y) \qquad ...1.10$$

which is a function defined on $\wp x \Re^N$ (with $\Re^N$ being the N-dimensional Euclidean space) such that for a given (s,y), its value depends on Y only through those i for iϵs. This brings to bear the desirable properties of a good estimator which include, unbiasedness, admissibility, efficiency as well as sufficiency. The common expectation of every researcher is that the desirable estimator be unbiased.

Basically, an estimator is unbiased for Y with respect to a sampling design **p** if

$$E_p(e(s,y)p(s) = Y \ \ \forall y \in \Re^N$$

$$\Rightarrow \sum_{s \in S}(e(s,y)p(s) = Y \qquad ...1.11$$

where E_p denotes expectation with respect to sample design (SD) **p** is true. When (1.11) is false, then the estimator under consideration is biased.

A combination of sampling design **p** and estimator $\hat{\tau}$ is called a strategy denoted by H(p, $\hat{\tau}$). H(p, $\hat{\tau}$) is also unbiased for Y if (1.10) holds true and its variance

$$V\{H(p,\hat{\tau}) = E(\hat{\tau} - Y)^2 \qquad ...1.12$$

Following the non-existence result, Godambe and Joshi(1965) developed admissibility criteria for an estimator $\hat{\tau}$ in the class **C** which is uniformly better than τ. This along with work by Basu(1971) opened up the modern scope of inference with respect to finite population, Rao(1966a) made a surprising revelation when he proposed alternative estimators in PPS sampling scheme that appeared to be more efficient than the conventional unbiased estimators even though the estimators were biased.

For this reason, we consider the Mean square error (MSE).

The MSE of $\hat{\tau}$ around Y with respect to sampling design **p** is

$$MSE(\hat{\tau}) = E(\hat{\tau} - Y)^2 = \sum_{s\in S}(\hat{\tau}(s,y) - Y)^2 p(s)$$

$$= E(\hat{\tau} - E(\hat{\tau}))^2 + (E(\hat{\tau}) - Y)^2$$

$$= V(\hat{\tau}) + \ (B(\hat{\tau}))^2 \qquad ...1.13$$

A sampling design SD(p) for an estimator $\hat{\tau}$ (say) is said to be better than another design SD(p’) in the sense of variance if variance of SD(p) is less than the variance of SD(p’) for another estimator $\hat{\tau}$, that is,

$$V_p(\hat{\tau}) \leq V_p(\hat{\tau}') \; \forall \; y \in \Re^N \qquad ...1.14$$

with strict inequality holding for at least one Y. This comparison is only possible when the sampling design is kept fixed. Similarly, a sampling strategy H(p, $\hat{\tau}$) is said to be better than another strategy H’(p’, $\hat{\tau}$’) in the sense of variance if

$$V\{H(p,\hat{\tau})\} \leq V\{H'(p',\hat{\tau}')\} \forall \; y \in \Re^N \qquad ...1.15$$

with strict inequality holding for at least one Y.

Next, we extend these strategies under the super-population (SP) model set up. In actual practise, information about the study population is not known. In the absence of this information, it is possible to utilize SP model to formalize the prior knowledge of the population under study.

Super-population model is usually imposed to give an idea about the relative performance of strategies appropriate to the model (Mukhopadhyay, 1996). Under this set up, a survey population is looked at as a random sample from super-population and inference is drawn about population parameter from a prediction theorist viewpoint.

Assume that $y=(y_1,y_2, ...,y_N)$ is a particular realization of a random vector $Y=(Y_1,Y_2, ...,Y_N)$ having a joint density ξ_θ indexed by a parameter vector $\theta = (\theta_1, \theta_2, \dots, \theta_k), \theta \epsilon \Theta$, parameter space, then ξ_θ belongs to a class of distribution $C=\{\xi_\theta\}$. Therefore, C is called a super-population model.

Now, given a sample S and corresponding y-values. Let $X=(x_1,x_2, ...,x_N)$, $x_i > 0$ be the corresponding auxiliary information, then ξ is usually modelled to reflect the auxiliary information so that one can estimate the unknown parameters and infer on the finite population.

Let the model based unbiased ξ -unbiased or estimator be defined as

$$\xi(\tau_s) = \tau, \forall\theta \in \Theta \text{ and } \forall_s\text{: p(s)} > 0 \qquad ...1.16$$

and ξ, υ and $\mathcal{C}$ denote the expectation, variance and covariance with respect to super-population distribution ξ,.

It is design –model based unbiased if

$$E\xi\left(\hat{\tau}_g\right) = \tau, \forall\theta \in \Theta. \qquad ...1.17$$

For comparing estimators under super-population model, the expected MSE is

$$\xi MSE\{(p,\hat{\tau})\} = E\xi(\hat{\tau} - Y)^2 \forall\ y \in \Re^N \qquad ...1.18$$

is best utilized when it is desirable to predict the total of the current population from which the sample has been drawn from. For comparison of estimator τ and τ', say, in terms of MSE, we have

$$\xi MSE\{(p,\tau)\} \leq \xi MSE\{(p,\tau')\} \forall\, y \in \Re^N \qquad ...1.19$$

In this work, the criterion for judgement under super-population model is the expected variance of an estimator under a given sampling strategy. Consequently, the role of the super-population model ξ is to choose between different strategies and has nothing to do with our final inference, which depends on the sampling design.

With respect to a given super-population model ξdefined on $\Re^N$, we shall define the followings:

$$\left.\begin{aligned} \mu_i &= \int y_i d\xi\,, & 1 \leq i \leq N \\ \sigma_i^2 &= \int (y_i - \mu_i)^2 d\xi, & 1 \leq i \leq N \\ \sigma_{i,j} &= \int (y_i - \mu_i)\left(\int (y_j - \mu_j\right) d\xi, & i \neq j = 1, \dots, N \end{aligned}\right\} \qquad ...1.20$$

So that the expected variance is

$$\nu = \xi V\{(p,\tau)\} = \int V(\tau) d\xi \qquad ...1.21$$

Let C denote the class of distributions ξ of Y satisfying the followings:

$$\left.\begin{aligned} &i. && \mu_i = \beta x_i, && 1 \leq i \leq N \\ &ii. && \sigma_i^2 = a x_i^g, \quad a > 0, g \geq 0, && 1 \leq i \leq N \\ &iii. && \sigma_{i,j} = 0, && i \neq j = 1, \dots, N \end{aligned}\right\}$$

where

$C = \sigma_{i.j}, \nu = \sigma_i^2$ and μ_i are the covariances, variances is the expected value respectively.

Smith(1938), Jessen(1942), Mahalanobis(1944) and Brewer(1963) have shown that the value of the parameter g lies between 0 and 2 as it relates to a sampling design.

The major interest in this study pertains to PPS WR or WOR sampling upon which the robustness of our estimator will be investigated in terms of the expected mean square error (ξMSE).

1.2 Use of Auxiliary Information in Surveys.

In survey sampling, information on a highly positively correlated auxiliary variable x with the study variable y is used to estimate the population parameter τ. most often, these information may be available in one form or the other and if used intelligently, it leads to the sampling strategies with higher efficiency compared to those in which no auxiliary information is used. The auxiliary information could take different forms for some population units. This could be in the form of parameters say, τ(y). Examples are, $\mu_x, C_x, CD(x)$ which are the population mean, coefficients of variation and determination of x respectively, and so on and this information could be known exactly or approximately.

Tripathi (1973, 1976) identified three ways in which auxiliary information could be utilized. These include;

i. At the pre-selection stage or design stage. Here, auxiliary information could be used for stratification or to form clusters;
ii. At the selection stage by use of probability proportional to size WR or WOR;
iii. At the post selection stage or estimation stage by using such estimators like ratio, regression, difference or product estimators for the population parameter of interest.

Higher precision could be achieved by using the auxiliary information for dual purposes of selection and estimation procedures (Tripathi, 1969, 1973).

In this study, auxiliary information giving rise to measure of size (or probability normed-size measure) is assumed at both the selection and estimation stage under linear regression, ratio, product and difference estimators as demonstrated in the works of Singh, Singh, Tailor and Allen(2002) and Singh and Tailor(2005).

As a slight deviation from the usual estimators, we considered the statistical distributional properties of a target population under linear regression model for which the intercept parameter is zero to generate the expectation of the study variable in the linear regression model and the expectation of the c^{th} standardized moment of the study variable given the measure of size variable.

These moments provide important information useful in specifying or defining an estimator. We define the coefficient of variation (*CV*), coefficient of determination (*CD*), coefficient of skewness (γ) and coefficient of kurtosis (*K*) as the parameter realized by these moments. These are defined as;

$$C_y = \frac{\sigma_y}{\mu_y}, \text{where } \sigma_y = \left[\frac{1}{N}\sum_{i=1}^{N}(y_i - \mu_y)^2\right]^{1/2}, \qquad ...1.22$$

$$\mu_y = \frac{1}{N}\sum_{i=1}^{N} y_i \qquad ...1.23$$

$$CD = \rho^2 = 1 - \frac{\sigma_\varepsilon^2}{\sigma_y^2},\ 0 \leq \rho^2 \leq 1 \qquad ...1.24$$

$$\gamma_y = \frac{E(y_i - \mu_y)^3}{\sigma^3} - 3 \qquad ...1.25$$

and

$$K_y = \frac{E(y_i - \mu_y)^4}{\sigma^4} \qquad ...1.26$$

Generally, the c^{th} central moment is defined by

$$E(y_i - \mu_y)^c = \int (y_i - \mu_y)^c\, p(y)dy \qquad c = 1,2,3,4 \qquad ...1.27$$

The essence of these moments is to provide a link between statistical properties enumerated above with the population correlation coefficient and by doing so, provide criteria for defining an estimator under the linear model.

1.3 Complex surveys

In order to estimate the parameters of a survey population like population total or mean, various sampling strategies have been developed premised on the kind of information required. Thus, in large-scale surveys, data on several characteristics of the study population are collected. Usually, selection of units from the population rarely involves just simple random sampling (SRS). Instead, more complex sampling schemes are employed to reflect complex underlying population structure. Most real life surveys employ the following features namely;

i. combination of sampling schemes;
ii. auxiliary or supplementary information which are known to assist in realizing a more efficient estimate of parameters when properly utilized;
iii. transformation of auxiliary variable used in calculating selection probabilities and utilizing the correlation between the study variables selection probabilities.

Unequal probability sampling (UPS) otherwise called probability proportional to size (PPS) sampling scheme is employed in complex surveys as it is suitable to designs and estimation of parameters in multiple character surveys.

Earlier works in complex surveys include that by Neyman(1934) on stratified random sampling, optimal allocation and logic of inference based on confidence intervals and Sukhatme (1935) on Pilot samples to implement Neyman allocation.

Conventional and the existing alternative estimators of population total have always assumed positive correlation between the study variables and the selection probabilities. However, it is known that correlation coefficient could also be negative in which case, there are a few literatures addressing this area.

Importantly, the existing estimators have always been assumed to be the best for all populations and conditions. This is not always the case following the non-existence of a uniformly most efficient estimator theory by Godambe(1955) and Basu(1971). For these reasons, this work is intended to utilize the available information about the study populations in order to develop alternative linear estimators in PPS sampling with replacement (WR) and without replacement (WOR) designs.

1.4 Alternative Estimators

The concept of "alternative estimators" denotes unconventional estimators or estimators that are different from the usual ones. The idea of alternative estimator is similar to that of hypothesis testing in which the usual hypothesis is the "null" hypothesis while the alternative hypothesis provides an option different from the usual one. Srivastava and Srivastava(2009) identified five standard senses of alternative hypothesis in which one population (or estimator in this case) is said to be located to the right of the other.

Specifically, suppose that the study variable of interest is y_i {i=1,2,3,...,N}. Let the measure of size variable be x_i {i=1,2,3,...,N} from which selection probabilities $p_i=x_i/X$ are derived. Let p_i* be the selection probabilities realized through certain transformation of p_i. Then, we can define a conventional estimator $\hat{\tau}_c$ and an alternative estimator say, $\hat{\tau}_{g,c}$ in terms of p_i and $p_{i,g}$* respectively. The hypothesis would be:

$$H_o: MSE(\hat{\tau}_c) = MSE(\hat{\tau}_{g,c})$$

against the alternative

$$H_1: MSE(\hat{\tau}_c) \neq MSE(\hat{\tau}_{g,c}) .$$

Certainly, if $MSE(\hat{\tau}_c) > MSE(\hat{\tau}_{g,c})$, then the alternative estimator would be preferred, where $MSE(\hat{\tau}_c)$ and $MSE(\hat{\tau}_{g,c})$ are mean squared errors of the conventional and alternative estimators respectively. It is worth to note that the estimators under comparison could all be biased or unbiased.

In this study, we draw inspiration from the works of Godambe(1955,1956), Rao(1966a, 1966b), Basu(1971), Amahia, Chaubey and Rao(1989), Grewal(1997) and Ekaette(2008) to develop alternative estimators when positive correlation between *y* and p_i exists and further insight from the contributions of Bedi(1995) and Bedi and Rao(1997) to develop alternative estimators with negative correlation coefficient between *y* and p_i.

1.5 Aim and Objectives of the study

The major aim of this study is to develop a class of alternative linear estimators for use in multi-character surveys.

The specific objectives include:

i. To propose generalized selection probabilities under linear framework for both negative and positive correlation between the study variables and selection probabilities by utilizing the c^{th}, (c = 1, 2, 3, 4) standardized moment of the study variable;
ii. Modifying the conventional Hansen and Hurwitz estimators for use under conditions of negative correlations;
iii. Utilize the proposed transformations to develop alternative linear estimators in PPSWR and PPSWOR designs;
iv. Investigate the consistency of some specified estimators under normal, theoretical distributions namely, normal, uniform, gamma and chi squared distributions.

1.6 Justification

The conventional estimator in PPSWR sampling and PPSWOR sampling schemes is useful only when it is assumed that the correlation coefficient between study variables and measure of size variables is positive. This is not always the case; correlation coefficient may be zero in the sample (as in Rao's estimator) or a negative quantity.

Previous studies on alternative estimators have defined PPS estimators with respect to positive correlation coefficients between the study and measure of size variables. However, these estimators are rigidly specified by the claim that a particular estimator is best for all study populations. Also, existing alternative estimators only consider the distribution of the target population with respect to the correlation coefficient ρ. In this work, we add the standardized moments in the study variables under linear framework.

Thus we developed a class of alternative linear alternative estimators that utilize moments in correlation coefficient and takes into cognisance the distributional properties of the survey population. The cases of negative and positive correlation coefficient between the study variables and selection probabilities are also investigated.

1.7 Significance of the study

The general class of alternative linear estimators defined by moments in correlation coefficient is intended to introduce flexibility in the definition of estimators for survey population. As every population possesses unique properties, their estimators would definitely have different specifications. This is further justified by the non-existence theorem of a uniquely efficient estimator for all populations due to Godambe(1955) and Basu(1958, 1971).

This procedure allows the use of precise estimators under different set-ups.

1.8 Scope and Limitation of the study.

Our study considered only linear transformations of the selection probabilities for use with homogenous linear estimators (HLE). It limits itself to uni-cluster and uni-stage sampling schemes. However, the findings of this thesis can be easily extended to those designs which are mostly applicable in large scale surveys.

1.9 The arrangements of this thesis

In addition to this introductory chapter which contains the various definitions and explanation of basic concepts used in the sequel, the present work contains five more chapters as follows:

In **chapter two**, we presented detailed review of existing literature that bothers on historical developments in sampling methodologies and hence, PPS sampling scheme. In other word, we showed the various developments from inceptions up to the point we are making our contributions with the aim of providing estimators that will depend largely on information obtained from moments of the target population.

In **chapter three,** we made some propositions leading to the development of the methodologies needed in this research. Firstly, we utilized the laws of direct and inverse proportions to propose transformations of selection probabilities in both cases of positive and negative correlation coefficients that will be needed. This is because;

they are pivot elements for defining PPS estimators. The transformed selection probabilities p_i were generalized under linear regression model and by this generalization, a link between correlation coefficient and the statistical properties defined by the first four standardized moments of the study variables was established so that we could estimate the desirable parameters of the study population.

By the methodologies above, we postulated that under the linear model, first order correlation coefficient ρ^1 was given by the ratio of the coefficient of variation of y and coefficient of variation of x; ρ^2 was linked with coefficient of determination; ρ^3 was linked with skewness while ρ^4 was linked with kurtosis. The range of the specification parameter c, was also defined.

In **chapter four,** we developed a class of linear estimators $\tau_{g,c}$ defined by the range of the specification parameter *c=1,2,3,4* and utilized them in the process of estimation and inference. This scenario was conducted under certain transformation for both cases of positive and negative correlation under PPSWR and PPSWOR designs.

We utilized the technique of Rao-Hartley and Cochran(1962) to study the relative efficiency of the estimators in this class at varying correlation for n > 2, specifically, n = 5. Similarly, the super-population model is utilized for comparing our estimators with some existing estimators that fall in this class as well as conventional estimators.

In **chapter Five**, we presented the various results of analysis for the four study populations with sample of sizes n = 2 and n = 5 for both sampling design and super-population model for the proposed and conventional estimators. We further investigated the consistency of the proposed estimators under some theoretical distributions namely, normal, uniform, gamma and chi-squared distributions. Thus, selection probabilities were simulated under normal, uniform, gamma and chi-square distributions and various estimates as well as their relative efficiencies were computed for both sampling design and super-population model for g = 0, 1, 2; ρ = 0, 0.1, 0.5, 0.9, 1 and the estimate of the correlation coefficient for the target population, $\hat{\rho}$. Comparison of estimates for realized by the class of linear estimators was also made.

Chapter six presented the summary of the major results reported in this work. The areas of possible future extension of works contained in this thesis are also included.

CHAPTER TWO

LITERATURE REVIEW

2.0 Introduction.

In this chapter, a review of related literature pertaining to unequal probability sampling is presented. We shall introduce various developments in survey sampling that led to the realization of multi-character survey and then, discuss various PPS sampling schemes with respect to their designs and estimation using direct responses.

2.1 Trends in Sampling

In the study of sample surveys and random experiments, Dalenius(1962) observes that the development of statistical theory and methods has grown tremendously in response to demands for tools to cope with the problem of uncertainty which arises when dealing with observations exhibiting variability. These studies are guided by the necessity of being able to measure the degree of uncertainty and the desire to regulate these uncertainty, which is a central problem in the theory and method of statistical inference.

The credit of placing "sample survey theory and method" within realm of random experiment is largely due to Neyman(1934) in whose paper marked the beginning of the concept of "probability Sampling". Madow(1948) obtained a result in probability sampling which was generalized. Turkey(1950) advanced the analytical tools for deriving higher moments using polykays as well as the derivation of moment coefficient of the k-statistics in the works of Wishart(1952).

The need to regulate the degree of uncertainty required the choice of criterion, that is, the measure of efficiency and the techniques for using it. Neyman(1934) introduced the criterion of minimizing the variance subject to fixed sample size. Yates and Zacopany(1935) gave a more general formulation of minimum variance (MV) subject to fixed cost and vice versa which has govern the design of large sample today. The area of inference have been explored by various scholars including Royall(1971a,1971b), Rao and Singh(1973), Royall and Cumberland(1981a,1981b), Das and Tripathi(1978) and Mukhopadhyay(1977,1978,1984,1991), and Godambe(1982).

The development of basic models meant for schemes used in survey also took various dimensions. Neyman(1934) formulated and solved the problem of the best allocation of sampling units among strata in stratified sampling. Jessen(1942) demonstrated the efficiency to be achieved in using a 'panel' when estimating changes in time. Hansen and Hurwitz(1943) extended the theory of sampling from finite population to cover complex designs and also introduced a scheme for multi-stage sampling using probability proportional to size to efficiently determined measure of size for selecting primary sampling units. Cochran(1946), Madow(1949), Yates(1948) developed the theory of systematic sampling.

The development of basic models follows some schemes of classification namely:

i. the simple random sampling group comprising the, Y, R and $\bar{Y}$ models as can be seen in the works of Basu(1958) among others;
ii. Stratified sampling which became very prominent owing to the problem of how best to stratify a population into a fixed number L, of strata. In this regards, progress have been made in determining optimum number of strata as well as developing computationally simple methods for approximating the exacts solution;
iii. systematic sampling as a solution to the problem of measuring the degree of uncertainty;
iv. Sampling n > 1 with unequal probabilities as a scheme developed by Hansen and Hurwitz(1943) which is characterized by sampling with replacement. The need for more efficient estimator gave rise to the use of sampling without replacement;
v. another aspect is the sampling scheme which selects a sample from the population to the sum of the measure of size with unequal probabilities which is often used in the selection of primary units in multi-state sampling scheme.

Our interest in this study is the estimation of sampling schemes under probability proportional to size otherwise, called PPS sampling.

2.2 Unequal probability sampling

Marriot(1990) defined the term Unequal probability sampling (UPS) as a method of sampling in which the units are selected with probability proportional to size (PPS) measure related to the characteristics under study.

Unequal probability sampling is either with replacement called Unequal Probability Sampling With Replacement (UPSWR) otherwise called Probability Proportional to size with Replacement (PPSWR) sampling or Unequal Probability sampling Without replacement (UPSWOR) otherwise called Probability Proportional to size Without Replacement (PPSWOR) sampling.

The theory of UPS with replacement was developed by Hansen and Hurwitz(1943). Prior to this development, there existed other sampling theory and practices as contained in the work of Neyman(1934) among others that assumed that the probability of selection within each stratum would be equal. Since then, considerable progress has been made with the contributions of many workers in the development of this area of study thereby, realizing tremendous progress over recent times.

Many works have been done covering the aspects of sample selection from a given universe and it has been shown that UPS provides more efficient estimator of population parameter than obtained from equal probability sampling. Thus, the theoretical framework by Hansen and Hurwitz(1943) otherwise called HH has become a cornerstone for the developments that sprang up in this area of study.

Madow(1949) proposed the use of systematic sampling with unequal probability to avoid the possibility of units being selected more than once. Midzuno(1950), and Narain(1951) considered the problem of sampling with varying probability without replacement. This was followed by Horvitz and Thompson(1952) who gave the theoretical background, Yates and Grundy(1953) and Sen(1953) who studied a more general method of sampling without replacement (WOR) and with varying probabilities, pointing out that the variance of the population parameters under Horvitz-Thompson estimator (HTE) is uniquely determined by the first and second order inclusion probabilities of units in the sample for a chosen design. Usually, the value of the auxiliary variable is chosen such that it is closely related to the study variable.

Attempts have been made to develop fixed sample size sampling designs with inclusion probabilities proportional to size (IPPS) measure called πPS design due to Hanurav(1967). Many sampling designs such as those due to Yates and Grundy(1953), Hanurav(1962), Fellegi(1963), Rao(1963), Hajek(1964), Caroll-Hartley(1964), Durbin(1967), Sampford(1967), Vijayan(1966,1968), Mukhopadhyay(1972), Sinha(1973), Sengupta(1981), Gupta, Nigam and Kumar(1982), Saxena, Singh and Srivastava(1986), Arnab(2001), Adhikary(2009), Alodat(2009) were developed using HTE.

Apart from sampling strategies consisting of πPS design and corresponding HTE, some other procedures of interest were developed. These include the Rao-Hartley-Cochran(1962) otherwise called RHC procedure, Midzuno(1950,1952) and Chikkagoudas(1967) which make use of several estimators other than HTE.

Procedures developed by Midzuno(1950,1952), Lahiri(1951), Sankarnarayana(1969), Despande(1978) gave unbiased estimation for ratio estimators. Mukhopadhyay(1972) and Sinha(1973) attempted to obtain sampling designs realizing a second order inclusion probabilities. This problem was also considered by Harzel(1986).

Das(1951), Raj(1956) and Murthy(1957) have suggested certain special estimator for use with YG(1953)'s draw-by-draw procedure. Works reviewing various sampling designs can be found in Brewer and Hanif(1983), Chaudhuri and Vos(1988) and Mukhopadhyay (1982).

Recent developments in the theory of PPS sampling have covered the areas of estimation involving indirect responses otherwise called, Randomized Responses (RR) whose estimation technique was developed by Warner(1965). The works of Arnab(1990), Chaudhuri and Adhikary(1990), Chaudhuri(2001a,2001b), Chaudhuri(2002), Chauhudri and Pal(2002), Sidhu, Bansal and Singh(2007), Chauhudri and Dihidar(2009) and Chauhudri(2010) have advanced the studies in this area as pertaining estimation of population proportions and totals.

2.3 Probability Proportional to Size With Replacement Sampling.

Sampling with replacement is described by Rao(1966a) thus " a fixed number of 'n' units is selected with replacement in contrast with simple random sampling without replacement where units are selected without replacement so that all the units in the sample are distinct.

Hansen and Hurwitz(1943) proposed the idea of sampling with probability proportional to size without replacement. Under this scheme, one unit is selected at each of the n-draws. For each i^{th} unit selected from the population, a selection probability is given as

$$p_i = \frac{x_i}{X}, \qquad ...2.1$$

where x_i is the measure for i^{th} population unit and $X = \sum_i^N x_i$.

Using the notations defined above, Hansen and Hurwitz(1943) gave the estimators of population total Y, as

$$\hat{\tau}_{HH} = \frac{1}{n}\sum_{i=1}^{N}\frac{y_i}{p_i} \qquad \ldots 2.2$$

whose estimator of the variance $V(\hat{\tau}_{HH})$ is

$$V(\hat{\tau}_{HH}) = \frac{1}{n}\sum_{i=1}^{N}\frac{1}{p_i}\left(y_i - p_i y_i\right)^2 \qquad \ldots 2.3$$

and the possible unbiased estimators of population variance are given as

$$V(\hat{\tau}_{HH}) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(\frac{y_i}{p_i} - \hat{\tau}_{HH}\right)^2 \qquad \ldots 2.4$$

and

$$V(\hat{\tau}_{HH}) = \frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n}\left(\frac{y_i}{p_i} - \frac{y_j}{p_j}\right)^2 \qquad \ldots 2.5$$

Rao and Hartley(1962) proposed a method for estimation of variance that always have smaller variance than the standard estimator in sampling with unequal probability and with replacement. This method does not entail heavy computation, even for a sample of n>2 and it enjoys the advantages of exact variance formula for any population size as compared with the asymptotic variance of Rao and Hartley(1962).

Rao(1978) looked at the robustness in large samples of the Hansen and Hurwitz strategy considering the population to be divided into two domains of sizes N_1 and N_2 in which case, the units in each domain obey the super-population model. Rao(1978) also compared the average biases of the two strategies assuming the size variable in the two domains to be independently and identically distributed gamma variable. Here, Rao(1978) concludes that the ratio estimator in SRSWR may perform better than the usual PPS estimator in PPSWR sampling scheme.

Royall and Herson(1973) and Godambe and Thompson(1977) considered specifically a situation in which the model failure of super-population model consists of latent order polynomial term in x or an intercept $\beta_{o.}$

Brewer(1979) pointed out that the Rao- Hartley(1962) use of stratified balances sampling scheme was a result that depended on the variance function in the primary model. Therefore, he proposed a combined estimation and selection scheme for use in large scale enterprise and establishment surveys. Brewer(1979) also demonstrated that his estimator is design-unbiased and subject to this constraint, has minimum expected variance under super-population model.

The advantage of Brewer's scheme over the Rao-Hartley's(1962) scheme is the removal of size stratification and it further allows for more general variance function thus, permitting a more efficient relationship between selection probability and unit size.

To enhance the efficiency of the HHE, Rao(1966a) introduced the idea of multiple characteristics and utilized the value of zero correlation coefficient in defining his estimator of population total.

Works ascertaining the validity of this estimator was carried out by Pathak(1966) and Rao(1993a,1993b). This introduced a new dimension in the study of PPS sampling schemes leading to various estimators by Bansal and Singh(1985), Amahia-Chaubey and Rao(1989), Kumar and Agarwal(1997), Grewal(1997), Ekaette

(2008), Singh and Horn(1998), Mangat and Singh(1993), Srivenkataramana (1980), Sahoo, Sahoo and Mohanty(1994) among others. These estimators shall be properly discussed in latter section under multi-characteristics.

2.4 Probability Proportional to size Without Replacement Sampling.

The concept of Unequal probability sampling without replacement was first used by Madow(1949) having utilized it with systematic sampling to avoid situation in which a unit is selected more than once.

Narain(1951) provided theoretical framework and selection procedure for this scheme which was fully developed by Horvitz and Thompson(1952) who suggested the estimator of population total, $\hat{\tau}_{HT}$, popularly called HTE for use with unequal probability sampling without replacement defined by

$$\hat{\tau}_{HT} = \sum_{i \epsilon S} \frac{y_i}{\pi_i} \quad \text{...2.6}$$

Whose variance was given as

$$V(\hat{\tau}_{HT}) = \sum_{i=1}^{N} \frac{(1-\pi_i)}{\pi_i} y_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{(\pi_{ij} - \pi_i \pi_j)}{\pi_i \pi_j} y_i y_j \quad \text{...2.7}$$

which Godambe and Joshi(1965) showed that under super-population model, the variance of Horvitz Thompson estimators attains lowed bound for any sampling design with the bound given as

$$E_M E_D (\hat{\tau}_{HT} - Y)^2 \geq \sum_{i=1}^{N} \sigma_i^2 \left(\frac{1}{\pi_i} - 1\right) \quad \text{...2.8}$$

Sen(1953), Yates and Grundy(1953) provided an alternative expression of the variance of Hansen and Hurwitz estimator as

$$V_{SYG}(\hat{\tau}_{HT}) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} \frac{(\pi_i \pi_j - \pi_{ij})}{\pi_{ij}} \left(\frac{y_i}{\pi_i} - \frac{y_j}{\pi_j}\right)^2 \quad \text{...2.9}$$

They showed that for n=2, it is a non-negative variance estimator. However, Sen(1953) have showed that for n = 2, $\pi_{ij} > \pi_i \pi_j$ for all i ≠ j when selection is made without replacement. Specifically, Vijayan(1968) identified the condition for which the HT and YSG estimators are unbiased and efficient, specifically, for a finite population consisting of N units and a positive valued auxiliary variable, taking the value of x_i and given a positive integer n satisfying $\max_i X_i \leq \frac{\sum_{i=1}^{n} X_i}{n} = \frac{X}{n}$ then we seek to find a sampling procedure satisfying the following conditions:

i. $\pi_i = \frac{nx_i}{X}$

ii. each sample contains n distinct units;

iii. $\pi_{ij} > 0$

iv. $\pi_{ij} \leq \pi_i \pi_j$

v. $\frac{\pi_{ij}}{\pi_i \pi_j} > \beta$ for β not too close to zero

Conditions (i) and (ii) above ensure the optimality of the sampling method in the Bayesian sense as proved by Godambe(1955) and Hajek(1958). Condition (iii) ensures the existence of unbiased variance estimator and (iv) ensures non-negativity of the SYG's estimator. Condition (v) ensures the stability of SYG's estimator.

Rao(1963) proved that under the Midzuno(1952) and SYG(1953) selection procedures for $\pi pswor$, the YSG estimator is always positive. Rao and Singh(1973) used the Brewer(1963) selection procedure to compare the HT and the YSG estimators for n = 2, employing a wide variety of population in which case, findings showed that the estimator of YSG is more stable than that of Horvitz and Thompson(1952). Brewer and Hanif(1983) and Shahbaz(2004) and Shahbaz and Hanif(2003) showed the same result.

The usual issue of concern in the application of the HT estimator is that the variance estimator of HT and that of SYG all require the computation of the joint inclusion probability, π_{ij}, and they are very difficult to apply especially as the computation of π_{ij} becomes very cumbersome. Several workers in the area of study have attempted to find approximations to the variance of HT in such a way that it does not involve the computation of π_{ij}'s. A simple approximation of π_{ij} in terms of π_i's and π_j's for selection procedure that ensures that $p_i = \frac{1}{2}\pi_i$, otherwise, $\pi_i = 2p$, is given by Brewer(1963), Rao(1965), Durbin(1967) and Sampford(1967) as,

$$\pi_{ij} = \frac{\pi_i \pi_j}{2 + \sum_{k=1}^{N} \frac{\pi_k}{1-\pi_k}} \left(\frac{1}{1-\pi_i} + \frac{1}{1-\pi_j} \right) \qquad ...2.10$$

Brewer and Hanif(1983) gave two approximation of π_{ij}'s:

$$\pi_{ij} = A\pi_i \pi_j + B(\pi_i + \pi_j) + C(\pi_i^2 + \pi_j^2) \qquad ...2.11$$

where $A = \frac{n^2}{n^2 - \sum_{j}^{N} \pi_j^2}$, $B = \frac{-n\sum_{j=1}^{N} \pi_j^2}{(n^2-2)[n^2 - \sum_{j}^{N} \pi_j^2]}$ and $C = \frac{n^2}{(n-2)[n^2 - \sum_{j}^{N} \pi_j^2]}$

Approximation II

This is given as

$$\pi_{ij} = (n-1) \frac{\sum_{r=0}^{\infty} \pi_i^{2r} \pi_j^{2r}}{\prod_{i=1}^{r} \sum^{r} \pi_k^{2r}} \qquad ...2.12$$

and it has been found that this approximation performs well even when a few values of π_{ij} are close to unity with each term being less than half of the proceeding one. The problem with this approximation is that it may not perform well when any π_i's is close to unity.

Harzel(1986) suggested another approximation of π_{ij} as

$$\pi_{ij} = \pi_i \pi_j - \frac{\pi_i(1-\pi_i) + \pi_j(1-\pi_j)}{N-2} + \frac{n\sum_{k=1}^{N} \pi_k^2}{(N-1)(N-2)} \qquad ...2.13$$

which may produce negative values of π_{ij}.

Hanif and Ahmad(2001) proposed another approximation to π_{ij} as

$$\pi_{ij} = \left(\frac{a_i + a_j}{2}\right)\pi_i \pi_j \qquad ...2.14$$

where a_i and a_j are carefully chosen in which case, they showed that

$$a_i = \frac{n-1}{n-\pi_i} \qquad ...2.15$$

so that when substituted in the variance formula of Horvitz and Thompson, we obtain

$$V(y'_{HT}) \approx \sum_{i=1}^{N} \pi_i (1 - \frac{n-1}{n-\pi_i}\pi_i \left(\frac{y_i}{\pi_i} - \frac{y}{n}\right)^2. \qquad ...2.16$$

Rao(1961) derived an expression for the variance of systematic sampling using the relation as

$$V(y'_{HT}) \approx \sum_{i=1}^{N} \pi_i (1 - \frac{n-1}{n_i}\pi_i \left(\frac{y_i}{\pi_i} - \frac{y}{n}\right)^2 - \frac{n-1}{n^2}\left(\sum_{i=1}^{N} 2\pi_i^3 - \frac{\pi_i^2}{2}.\sum \pi_j^2\right)\left(\frac{y_i}{\pi_i} - \frac{y}{n}\right)^2 + \frac{2(n-1)}{n^3}\left(\sum_{i=1}^{Nr} \pi_i y_i - \frac{y}{N}.\sum \pi_j^2\right)^2 \qquad ...2.17$$

which is accurate to order N^0. Rao(1961) further showed that the asymptotic variance formula to order N^0 for a sample (n=2) is

$$V(y'_{HT}) \approx \sum_{i=1}^{N} \pi_i (1 - \frac{\pi_i}{2})\left(\frac{y_i}{\pi_i} - \frac{y}{2}\right)^2 - \frac{1}{2}\left(\sum_{i=1}^{N} \pi_i^3 - \frac{\pi_i^2}{4}.\sum^{N} \pi_j^2\right)\left(\frac{y_i}{\pi_i} - \frac{y}{2}\right)^2 + \lambda\left(\sum_{i=1}^{N} \pi_i y_i - \frac{y}{2}.\sum \pi_j^2\right)^2 \qquad ...2.18$$

where λ =3/32 for Narain(1951) procedure, 1/8 for Caroll-Hartley(1964) repetitive procedure and ¼ for random systematic procedure of Goodman and Kish(1950).

Rao(1963) further showed that the approximate formula of order n^1 for a sample of size n is

$$V(y'_{HT}) = \sum_{i=1}^{N} \pi_i (1 - \frac{n-1}{n_i} \pi_i \left(\frac{y_i}{\pi_i} - \frac{y}{n} \right)^2 \qquad ...2.19$$

The work by Shahbaz(2004), Sahoo, Mishra and Senapati(2005), Senapati, Sahoo and Mishra(2006), Adhikary(2009) bothered on the improvement of PPS estimators and so, developed new estimator of population total following the method of Horvitz and Thompson(1952), Murthy (1957) and the Durbin (1967) selection procedure. Efficiency of the new estimator was compared with various existing estimators for a sample of size 2 and also derived a Design-based and Model-based variance and found out that Model based variance achieved the Godambe-Joshi lower bound. There are several other developments in this area of variance estimation which could further be exploited.

2.5 Estimators in PPS sampling with replacement for multiple characteristics.

Studies involving estimation of population parameters relating to several population characteristics gave rise to the use of auxiliary variable which provides a measure of size for selecting a sample of units with PPS sampling scheme. Here, it might happen that some of the study variables are poorly correlated with the selection probabilities used for PPS sampling scheme while some may be highly correlated with the study variable. It is also possible that the dimension of the correlation coefficient could be positive or negative.

The work of Hansen and Hurwitz(1943) and propositions by Mahalanobis(1944) and Godambe(1955) prompted further studies in PPS sampling scheme and hence the developments in this area of knowledge. Work on the analysis of dispersion in sample survey involving multiple characteristics was done by Chakravarti(1954) with less emphasis on estimation of parameters of interest. However, there was remarkable development in the area of estimation and inference in the following decades. This was again prompted by the need for design and analysis of complex surveys as in Hajek(1958), Hartley and Rao(1962), Chikkagourdas(1967) and Hanif and Brewer(1980) among others in order to firstly address the problems of developing economies in the 1960's and 1970's and secondly, develop the theory of survey sampling.

Rao(1966a) in his study of the number of chickens in a farm (Y) and the farm size (X) being the auxiliary variable saw the variables of study to be poorly correlated. Thus, he proposed an alternative estimator when the study variable and the auxiliary variable are unrelated defined as

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{N}\frac{y_i}{p_i^*}, \qquad \text{...2.20}$$

where

$$p_i^* = \frac{1}{N}.$$

For this, he demonstrated that the estimators though biased, are likely to have smaller Mean square error (MSE) than the corresponding conventional unbiased estimator particularly in small samples. Again it is argued that the bias of such estimators are same for all sample sizes unless the study variable and the auxiliary variable are uncorrelated, in which case, they are unbiased.

Bansal and Singh(1985, 1989, 1990) observed that the circumstance considered by Rao(1966a) is not a common occurrence in the real life since population correlation between the study and auxiliary variable is never exactly zero. Thus Bansal and Singh(1985) suggested an alternative estimator to cater for the correlation that might have existed between the study and auxiliary variable. Here, it is assumed that the variables under consideration are poorly correlated. Again, this estimator comprises a transformation that is non-linear in nature and is assumed efficient for all populations. To appreciate the beauty of this work, we specify the linear estimator of the form:

$$\hat{\tau} = \sum_{i=1}^{N} b_{si} I_{si} y_i \qquad \text{...2.21}$$

where $I_{si} = \begin{cases} 1, \ if\ i \in s \\ 0, if\ i \in s' \end{cases}$ and bsi is weight not dependent on yi but is design specific,

Thus the estimator proposed by Bansal-Singh(1985) is given as,

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{N}\frac{y_i}{p_i^*}, \quad p_i^* = (1 + 1/N)^{1-\rho}(1 + p_i)^{\rho} - 1 \qquad ...2.22$$

Amahia *et al*(1989) observed that the work of Bansal and Singh's(1985) made mention of bias of the new estimator which was expected to be smaller than Rao(1966a) but did not derive any expression for bias or compare them. They further argue that it is not quite usual to assume that the expected value of the residual variance takes a well known form while using super-population model as used by Bansal and Singh(1985). Again, it is observed in Amahia(1989) that the use of p_i^* is without motivation except that it reduces to P_i when $\rho=1$ and 1/N when $\rho=0$. Importantly, the values of p_i^* are in some cases, negative and in most cases, do not sum to unity.

On this note, Amahia *et al*(1989) provided simpler alternative estimators of the population total when there is positive correlation between the study and auxiliary variable. One of such estimator is:

$$\hat{\tau} = \frac{1}{n}\sum_{i=1}^{N}\frac{y_i}{p_i^*}, \, p_i^* = \frac{1-\rho}{N} + \rho p_i \qquad ...2.23$$

satisfying all the boundary conditions of a probability normed-size-measure.

Grewal(1997)'s estimator mimics that of Amahia but with the transformation

$$p_i^* = \frac{1-\rho^{1/3}}{N} + \rho^{1/3} p_i \qquad ...2.24$$

and observed that in some cases, it performed better than the Amahia et al's(1989) estimator.

Singh and Horn(1998) also proposed an alternative estimator for estimating population totals in multi-character survey sampling when certain variables have poor positive correlation and others have poor negative correlation with selection probabilities. They showed that the estimators proposed by Hansen and Hurwitz (1943), Rao (1966), Singh, Singh and Shukla(1993) and Sahoo et al. (1994) are special cases of the proposed estimator.

Singh and Tailor(2003, 2005) suggested series of estimators of population totals under certain transformations of selection probabilities among which include the followings:

$$p_{i,1}^{*} = \left(1+\frac{1}{N}\right)^{(1-\rho)(1+\rho)} (1+p_i^{+})^{\rho(1+\rho)/2}(1+p_i^{-})^{-\rho(1-\rho)/2}\left(\frac{1}{N}\right)^{(1-\rho)(1+\rho)}-1 \quad ...2.25$$

$$p_{i,2}^{*} = \frac{(1-\rho)(1+\rho)}{N} + \frac{1}{2}[\rho(1+\rho)p_i^{+} - \rho(1-\rho)p_i^{-}] \quad ...2.26$$

where

$$p_i^{+} = \frac{x_i}{X}; X = \sum_{i=1}^{N} x_i \quad ...2.27$$

and

$$p_i^{-} = \frac{z_i}{X}; X = \sum_{i=1}^{N} x_i \quad ...2.28$$

with

$$z_i = \frac{X - nx_i}{N-n}; X = \sum_{i=1}^{N} x_i \quad ...2.29$$

The transformations combined two forms of selection probabilities as well as two dimensions of correlation in a single scheme. They are non-linear in nature and hence complex. Another estimator proposed by Singh, Grewal and Joarder(2004) has the transformation defined by

$$p_i^{*} = (p_i^{+})^{\rho(1+\rho)/2}(p_i^{-})^{-\rho(1-\rho)/2}\left(\frac{1}{N}\right)^{(1-\rho)(1+\rho)} \quad ...2.30$$

They also proposed a general class of finite population parameter estimators in multi-character survey and showed that the proposed estimator by Bansal and singh(1985) and Amahia et al(1989) are special cases of the general class of estimators for PPSWR strategy. The general class based on Taylor's approximation, defined the estimator as:

$$\hat{\tau}_g = \frac{1}{n}\sum_{i=1}^{N} y_i\,(H(P_i)) \quad ...2.31$$

with variance

$$v(\hat{\tau}_g) = \frac{1}{n}[\sum_{1}^{N} y_i^2 p_i (H(p_i))^2 - (\sum_{i=1}^{N} y_i\, p_i H(p_i))^2] \quad ...2.32$$

with $H(p_i)$ satisfying the regularity conditions.

Ekaette(2008) observed that the claim by Singh, Grewal and Joarder(2004) of having the estimator by Bansal and Singh(1989) in the generalized linear class of estimators is not true as the transformation used by them does not always satisfy all the boundary conditions of normed size measure namely, 0<pi<1 and $\sum_{i}^{N} p_i = 1$. Therefore, Ekaette(2008) proposed an alternative estimator defined by

$$\hat{\tau}_{g,\alpha} = \frac{1}{n}\sum_{i=1}^{N} \frac{y_i p_i}{p_i^{\alpha}} \qquad ...2.33$$

with

$$p_i^{\alpha} = \frac{1-\rho^{\alpha}}{N} + \rho^{\alpha} p_i,$$

where ρ^{α} satisfying all the boundary conditions.

Recent developments include the works of Gajendra, Singh and Singh(2010) that considered multi-auxilliary variables and post-stratification.

2.6 PPS estimators under negative correlation.

When the auxiliary variable x is negatively correlated with the study variate y, Robson(1957) proposed the product estimator of the population mean or total and was further developed by Murthy(1964).

Since then, a lot of development have been made in this area of product estimator including the works of Srivenkatarmana(1980), Singh(1986), Menedez and Ferrales (1989), Agrawal and Jain(1989) and Sahoo(1995). Studies on multivariate product estimators that deal with auxiliary information include the work of Olkin(1958), Singh(1967), Lui(1990), Agarwal and Panda(1993), Singh, Singh and Shukla(1993) proposed a general class of product type estimator under super-population model and also multivariate product estimators.

Works that utilized the coefficient of variation in estimating the finite population total include that of Das and Triparthi(1980), Sisodia and Dwivedi(1981), Singh and Upadhyaya(1986), Pandey and Dubey(1988), Singh and Singh(1998), Singh and Taylor(2005), Singh and Kumar(2009) among others.

Scholars such as Sahoo, Sahoo and Mohanty(1994), Bedi(1995), Bedi and Rao(1997), Singh and Horn(1998), Sahoo(1995), Sahoo, Mishra and Senapati(2005), Sahoo, Das and Singh(2006) and Sahoo, Senapati and Mongaraj(2010) worked on negatively correlated characteristics with complex transformations of the selection probabilities with little applications to PPS sampling schemes. Thus, the question on the efficiency of the estimators under PPS sampling schemes was not properly addressed.

To devise an estimator in PPS sampling scheme when it is apparent that the regression slope indicates inverse relationship between the study and measure of size variables we shall show that Hansen and Hurwitz's(1943) estimator only requires a little modification under certain law of variation to realize an estimator for negatively correlated variables.

2.7 Moments in correlation coefficient

Classical regression estimation makes the assumption of normality of model components and based on this assumption, estimation is made. This assumption does not always hold as in most cases, researchers involved in empirical researches deal with samples drawn from population and apart from being highly variable, are non-normal. Conventional and alternative estimators in probability proportional to size (PPS) sampling utilized the correlation structure based on this assumption without regards to assumption failure. It is based on this note that we examine the failure and suggest alternative estimators that will be applied under assumption failure.

Dodge and Rousson, (2000,2001) showed that, in the context of linear models, the response variable will always have less skew than the explanatory variable and this also applies to the kurtosis of the two variables. These facts can be used to determine the direction of dependence specifically, using third and fourth order moments, and information concerning the deviation of variables from normality. Thus modelling the variables is sensitive to various data distributions, sample size and simple correlation structure. Other workers in this area include Muddapur(2003), Shimizu and Kano(2006), Sungur(2005), Rovine and von Eye (1997), Rodgers and Nicewander, (1988).

We shall show the theoretical relationship between correlation coefficient and other statistical properties such as coefficients of variation, determination, skewness and kurtosis from the statistical moment perspective. Here, we draw inspiration from works of Roger and Nicewander(1988), Dodge and Rouson(2000), Sungur(2005), Dodge and Yadegari(2009) to establish the theoretical relationship between correlation coefficient and other statistical properties that are related by the model

$$y = \alpha + \beta x + \varepsilon \qquad ...2.34$$

where ε is an error random variable independent of x. The model coefficients are α and β which are model intercept and slope respectively. In ratio estimation, it is usually assumed that the intercept parameter is zero. The covariance between x and y is given by

$$Cov(x, y) = E[(x - E[x])(y - E[y])] = \beta\sigma_x^2 \qquad ...2.35$$

and the correlation coefficient is

$$\rho_{x,y} = \frac{Cov(x,y)}{\sigma(x)\sigma(y)} = \beta\frac{\sigma_x}{\sigma_y} \qquad ...2.36$$

where σ_x and σ_y are standard deviation of x and y respectively.

Let y be the response variable and x be the explanatory variable. Then, the skewness of x and the skewness of y are defined by the third moment, that is,

$$\gamma_x = \frac{E(x-E(x))^3}{\sigma_x^3} = \frac{\mu_{3,x}}{\sigma_x^3} \qquad ...2.37$$

and

$$\gamma_y = \frac{E(y-E(y))^3}{\sigma_y^3} = \frac{\mu_{3,y}}{\sigma_y^3} \qquad ...2.38$$

respectively.

The fourth moment, the kurtosis of x and y, is

$$K_x = \frac{E(x-E(x))^4}{\sigma_x^4} = \frac{\mu_{4,x}}{\sigma_x^4} \qquad ...2.39$$

and

$$K_y = \frac{E(y-E(y))^4}{\sigma_y^4} = \frac{\mu_{4,y}}{\sigma_y^4} \qquad ...2.40$$

respectively. We assume that x, y .≈ N(0; 1) with $\gamma = 0$ and K = 3.0 where γ and K are coefficients of skewness and kurtosis respectively.

CHAPTER THREE

GENERALIZATION OF SELECTION PROBABILITIES IN PPS SAMPLING DESIGN.

3.1 Introduction

In order to develop alternative estimators in probability proportional to size (with or without replacement) sampling designs taking cognizance of the moment characteristics of the target populations as well as correlation coefficient, appropriate transformations of selection probabilities are developed. The generalized transformation incorporated moments in correlation coefficient between study and measure of size variables so as to provide the required probability normed-size measure. In essence, appropriate specification of the selection probabilities is a prelude to estimation in PPS sampling.

In this section, we shall show that ratio estimator is a consequence of positive correlation between the study variables and selection probabilities. Next, we present the generalization of the resulting probability normed-size measures. Furthermore, we shall propose a transformation of selection probabilities under law of inverse proportion. The transformation is proposed for use when negative correlation between the study and auxiliary variables is encountered. This leads to the modification of the conventional estimators, namely, Hansen and Hurwitz Estimator $(\hat{\tau}_{HH})$ or $(\hat{\tau}_c)$ or simply HHE in the case of PPSWR sampling design and Horvitz and Thompson Estimator $(\hat{\tau}_{HT})$ or simply HTE under PPSWOR or πPS sampling design.

To achieve this objective, a link between correlation coefficient and statistical properties namely, coefficients of variation, determination, skewness and kurtosis will be established under linear framework which will, to a larger extent, help in the specification of the PPS estimators.

3.2 Transformation for selection probabilities under positive correlation.

In this section, we begin by specifying the homogenous linear estimator of the form:

$$\hat{\tau} = \sum_{i=1}^{N} b_{si} I_{si} y_i, \qquad \text{...3.1.1}$$

where

$$I_{si} = \begin{cases} 1, \ if \ i \in s \\ 0, if \ i \notin s \end{cases}$$ is an indicator variable

and

b_{si} is the weight not depending on y_i but on the sampling design. $\hat{\tau}_c$ is the conventional estimator of population total.

Hansen and Hurwitz(1943) defined this estimator as

$$\hat{\tau}_{HH} = \frac{1}{n} \sum_{i \epsilon \Omega} \frac{y_i}{p_{i,g}^*} \qquad \text{...3.1.2}$$

where p_i* is the transformed selection probabilities. In the case of Hansen and Hurwitz Estimator (HHE), $p_{i,g}* = p_i$; $p_i = x_i/X$ and $X = \Sigma x_i$.

Rao(1966a) provided an alternative estimator of population total, $\hat{\tau}_R$ which is useful when the study variable and the selection probabilities are unrelated. The transformation for this estimator is defined as

$$p_i^* = \frac{1}{N} \qquad \text{...3.1.3}$$

where N is the Population size of both the study and auxiliary variables.

Bansal and Singh(1985) proposed an estimator $\hat{\tau}_{BS}$, whose transformation is given as,

$$p_i^* = (1 + 1/N)^{1-\rho}(1 + p_i)^{\rho} - 1 \qquad \text{...3.1.4}$$

where ρ is the population correlation coefficient.

Amahia, Chaubey and Rao(1989) proposed alternative estimators $\hat{\tau}_{ACR}$ for estimating population total when there exist positive correlation between the study variables and selection probabilities. The transformation required by this estimator is given as;

$$p_i^* = \frac{1-\rho}{N} + \rho p_i \qquad ...3.1.5$$

which satisfied boundary conditions of probability normed-size measures.

Other scholars in this area include Grewal (1997) whose estimator $\hat{\tau}_G$, mimics the work by Amahia but with the transformation

$$p_i^* = \frac{1-\rho^{1/3}}{N} + \rho^{1/3} p_i \qquad ...3.1.6$$

and discovered that in some cases, this estimator performed better than the Amahia *et al's*(1989) estimator.

Ekaette(2008) proposed an alternative estimator $\hat{\tau}_E$, whose transformation is of the form;

$$p_i^\alpha = \frac{1-\rho^\alpha}{N} + \rho^\alpha p_i \qquad ...3.1.7$$

where ρ^α satisfying all the boundary conditions of probability normed-size measures.

We note here that the use of α by Ekaette(2008) in ρ^α is without justification as earlier, Grewal(1997) have used α=1/3 with p_i^α satisfying the boundary conditions. However, the view of Ekaette(2008) pointing at Bansal and Singh's(1985) estimator as not a member of this class holds true as the transformation utilized by the authors is non-linear in p_i under a linear framework.

Secondly, the use of the super population parameter α in the function, p_i^α is not very appropriate as minimum variance is always attained at α=[0.1] under the super-population model and this exaggerates the range of α defined by the Ekaette's estimator. We posit that α should be thought of as moments in correlation than a parameter in the super-population model. Thus, $p_i^\alpha = f(N, \rho^\alpha, p_i)$, i=1,2, ..., N, αϵR>0 is a consequence of statistical properties which makes $p_i^\alpha = f(N, \rho^\alpha, p_i)$ a function in p_i and also a function in ρ^α thereby, giving rise to the α^{th} ordered moments in ρ.

For convenience, let α be defined as c and $p_i^* = f(N, \rho^\alpha, p_i)$ then the behaviour of ρ^c can be ascribed moment in ρ, which can also be linked to the behaviour of coefficients of determination, skewness, kurtosis or coefficient of variation.

Definition 3.1: Consider a finite population Ω of N identifiable units uniquely labeled i=1,2, …, N on which are defined two real valued variables y and x assuming $y_i(>0)$ and $x_i(>0)$. Let a sample of size n be drawn with replacement from Ω and we suppose that y and x are positively correlated, then the conventional estimator is the ratio estimator especially when the regression line passes through the origin.

To justify this definition, we are establishing that the ratio estimator results from the law of direct proportion.

Theorem 3.1: Let $y \propto x$ or p be such that y and x or p are positively correlated. As a consequence of this relationship, the estimator of population total is $\hat{\tau}_{HH} = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{p_i}$, which is the Hansen and Hurwitz's estimator, $\hat{\tau}_{HH}$.

Proof: Let $y_i \propto x_i$ or p_i, then

$$y_i = \tau p_i$$

and

$$\tau = y_i / p_i.$$

Taking summation on both sides over the sample, we have

$$n\tau = \sum_{i=1}^{n}\frac{y_i}{p_i}$$

so that

$$\tau = \frac{1}{n}\sum_{i=1}^{n}\frac{y_i}{p_i} = \hat{\tau}_{HH}. \qquad \ldots 3.1.8$$

Theorem 3.2: If $y \propto x$ or p with $\hat{\tau}_R = \frac{y_i X}{x_i}$ defining the classical ratio estimator, then the required transformation for the selection probabilities is

$$p_i = \frac{x_i}{X} \qquad \text{…3.1.9}$$

Proof: If $y \propto x \Rightarrow y \propto p$, then

$$p_i = \frac{x_i}{X}.$$

This is also evidenced in $\hat{\tau}_R = \frac{y_i X}{x_i} = y_i p_i^{-1}$ when size measure is being considered so that $p_i^{-1} = \frac{X}{x_i}$ or $p_i = \frac{x_i}{X}$.

3.3 Generalized Linear Transformation under Positive correlation.

Proposition 3.1: The function expressing the relationship when y and x or p are positively correlated is of the form:

$$p_{i,g}^{*} = \beta_1 x_i + \varepsilon_i \qquad \text{…3.1.10}$$

Proof: expression 3.1.10 is a special case of linear regression model which is linear in x and hence, p_i.

It is obvious that 3.1.10 is useful when β_1 and hence, ρ is positive.

Proposition 3.2: Let $f(N, \rho^c, p_i) = \beta_1 p_i + \varepsilon_i$ so that y and p are directly proportional. Let p_i (i = 1,2,…,N) be a set of selection probabilities with $\sum_{i \in \Omega} p_i = 1$ and let there be a function say, f : N*[0,1]*[0,1] → [0,1], then, f must be a function satisfying the following regularity conditions:

i. $f(N, 0, p_i) = \frac{1}{N}, \forall\, 1 \leq i \leq N, N \in \mathbb{N};$
ii. $f(N, 1, p_i) = p_i, \forall\, 1 \leq i \leq N, N \in \mathbb{N};$
iii. $0 < f(N, \rho^c, p_i) < 1, \forall\, 1 \leq i \leq N, N \in \mathbb{N},\ c > 0,\ \rho > 0;$
iv. $\sum_{i=1}^{N} f(N, \rho^c, p_i) = 1,\ \ 0 < \rho^c < 1, N \in \mathbb{N},\ c > 0,\ \rho > 0$

We shall justify the propositions above by theorem 3.3 below.

Theorem 3.3: If f is uniformly continuous in p_i, and fulfils the regularity conditions (i) – (iv) above, then,

$$f(N, \rho^c, p_i) = \big(1 - g(c)\big)N^{-1} + g(c)p_i, 0 \le g(c) \le 1, 0 < p_i < 1, g(c) = \rho^c.$$

Proof: Let 1 ≤ i ≤ j ≤ k ≤ N, i ≠ j ≠ k be fixed points and let $p_{k,\epsilon} = p_{k-\epsilon}$, $p_{j,} = p_{j,}$, $p_{i,\epsilon} = p_{i+\epsilon}$, then from (iv) for p_k, k=1,2, …, N and $p_{k+,\epsilon}$, k=1,2,…,N, we have

$$\textstyle\sum_{i=1}^{N} f(N, g(c), p_k) = 1 \text{ and } \sum_{i=1}^{N} f\big(N, g(c), p_{k,\epsilon}\big) = 1 \qquad \ldots 3.1.11$$

So that

$$\textstyle\sum_{i=1}^{N} f(N, g(c), p_k) - \sum_{i=1}^{N} f\big(N, g(c), p_{k,\epsilon}\big) = 0 \qquad \ldots 3.1.12$$

$$\Rightarrow \textstyle\sum_{i=1}^{N} f(N, g(c), p_i) - \sum_{i=1}^{N} f(N, g(c), p_{i+\epsilon}) + \sum_{i=1}^{N} f(N, g(c), p_k) - \sum_{i=1}^{N} f(N, g(c), p_{k-\epsilon}) = 0 \qquad \ldots 3.1.13$$

Dividing (3.1.13) by ϵ and taking limit as ϵ→0, we have

$$\frac{d}{dp} f(N, g(c), p_{i)} = \frac{d}{dp} f(N, g(c), p_{k)} \qquad \ldots 3.1.14$$

Now, fixing p_i as constant and varying p_j's in (3.1.14), we get

$$\frac{d}{dp} f(N, g(c), p) = g(c) * f(p) \qquad \ldots 3.1.15$$

Now, integrating (3.1.15) we get

$$\int \frac{d}{dp} f(N, g(c), p) d(p) = g(c) * \int f(p) d(p) = C_1 + g(c) * p_i \qquad \ldots 3.1.16$$

Taking summation over (3.1.16) we get

$$1 = N * C_1 + g(c) * \textstyle\sum_{i=1}^{N} p_i$$

For non-negativity, $0 \le g(c) \le 1$ and hence, $N^* C_1 + g(c) = 1$.

Therefore,

$$C_1 = (1 - g(c))N^{-1} \qquad \ldots 3.1.17$$

which depends on N and ρ^c.

Since our task is to select the term $g(c)$ as a function of c, then we choose $g(c) = \rho^c$ so that

$$f(N, \rho^c, p_i) = p_{i,g}^* = \frac{1-\rho^c}{N} + \rho^c p_i \qquad \ldots 3.1.18$$

with $0 \leq \rho^c \leq 1$.

Expression 3.1.18 is the generalized transformation required for defining the class of linear estimators in PPS sampling scheme.

It is clear from the expression above that by

i. condition (i) above , when $\rho = 0$ and $c = 0$ then $g(c) = \rho^c = 0$;

ii. condition (ii) above, when $\rho = 1$ and $c \geq 0$, then $g(c) = \rho^c = 1$

iii. condition (iii) above, when $0 < \rho^c < 1$ and $c \neq 0$, we have

$$f(N, \rho^c, p_i) = \left(1 - g(c)\right)N^{-1} + g(c)p_i = p_{i,g}^* = \frac{1-\rho^c}{N} + \rho^c p_i \qquad ..3.1.19$$

which concludes the proof.

3.4 Transformation for selection probabilities under negative correlation.

In this section, a transformation to further the work on linear estimators in PPS sampling when the study variables and size-measures are negatively correlated is proposed. We also show that under homogenous linear estimator in which the study and auxiliary variables are inversely proportional, the required estimator is obtained by transforming the measure of size variables x.

Definition 3.2: Consider a finite population Ω of N identifiable units uniquely labeled 1,2, …, N on which are defined two real value variables y and x assuming $y_i(>0)$ and $x_i(>0)$. Let a sample of size n be drawn with replacement from Ω and we suppose that *y* and *x* are negatively correlated, then the conventional product estimator is defined by

$$\hat{\tau}_p = \frac{y_i x_i}{X} \qquad \ldots 3.1.20$$

Direct transformation of selection probabilities in the case of 3.1.20 above will yield an estimator that is meaningless even though it possesses the properties expected of an inverse relationship between y and x. Thus, our desire in this study is to modify the HHE which permits the use of a measure of size instead of the conventional product estimator. We consider the theorems below:

Theorem 3.4: Let $y \propto 1/x$ such that y and x or p are negatively correlated, then the transformation for the selection probabilities required here is

$$p_i = \frac{z_i}{Z}$$

where

$$z_i = \frac{1}{x_i} \text{ and } Z = \sum_{i=1}^{N} z_i.$$

Proof: If y $\propto$ 1/x $\Rightarrow$ y $\propto$ z, where z = 1/x. Then, $p_i = \frac{1/x_i}{\sum_{i=1}^{N} 1/x_i} \Rightarrow p_i = \frac{z_i}{Z}$

Remark 3.1.1: Under the transformation above, p_i is the selection probabilities realized for a relationship that is inversely proportional. It will be sufficient to utilize the PPS estimator to obtain the estimate of population characteristics instead of the conventional product estimator because the transformation has changed the correlation coefficient from negative to positive.

Remark 3.1.2: We shall call these selection probabilities as probability Proportional to Z, otherwise, PPZ and the corresponding estimator as the Modified Hansen and Hurwitz Estimator (MHHE).

Remark 3.1.3: This transformation has the properties of harmonic mean

Remark 3.1.4: This transformation can be utilized in 3.1.19, that is, the generalized transformation for estimating population characteristics of interest.

3.5 Relationship between ρ and other statistical properties.

Now, we show the links between correlation coefficient and other statistical properties such as coefficients of variation, determination, skewness and kurtosis based on expectation of the linear regression model as well as the expectation of the c^{th} standardized moments of the study variable in the linear regression model below.

Proposition 3.3: Consider the linear model

$$y = \beta x + \varepsilon \qquad ...3.1.23$$

Where y is the response variable, x is the explanatory variable, β is the slope parameter and ε is the error term, Then, the expected value of the c^{th} standardized moment of the study variable is given by

$$E\left(\frac{y-\mu_y}{\sigma_y}\right)^c = E\left[\beta\left(\frac{x-\mu_x}{\sigma_y}\right) + \left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_y}\right)\right]^c, c = 1,2,3,4 \qquad ...3.1.24$$

Proof:

From 3.1.23, we have

$$y - \mu_y = \beta(x - \mu_x) + (\varepsilon - \mu_\varepsilon) \qquad ...3.1.25$$

Standardizing 3.1.25 above, we obtain

$$\frac{y-\mu_y}{\sigma_y} = \frac{\beta(x-\mu_x)}{\sigma_y} + \frac{(\varepsilon-\mu_\varepsilon)}{\sigma_y}$$

$$= \frac{\rho\sigma_y(x-\mu_x)}{\sigma_x\sigma_y} + \frac{\sigma_\varepsilon(\varepsilon-\mu_\varepsilon)}{\sigma_\varepsilon\sigma_y}$$

$$= \frac{\rho(x-\mu_x)}{\sigma_x} + \left(\frac{\sigma_\varepsilon}{\sigma_y}\right)\frac{(\varepsilon-\mu_\varepsilon)}{\sigma_\varepsilon}$$

The c^{th} moment of the standardized variable *y* is:

$$\left(\frac{y-\mu_y}{\sigma_y}\right)^c = \left[\frac{\rho(x-\mu_x)}{\sigma_x} + \left(\frac{\sigma_\varepsilon}{\sigma_y}\right)\frac{(\varepsilon-\mu_\varepsilon)}{\sigma_\varepsilon}\right]^c$$

whose expectation is

$$E\left(\frac{y-\mu_y}{\sigma_y}\right)^c = E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right) \quad + R_{\sigma_{\varepsilon,y}}\left(\frac{\in-\mu_\varepsilon}{\sigma_\varepsilon}\right) \quad \right]^c$$

where $R_{\sigma_{\varepsilon,y}} = \left(\frac{\sigma_\varepsilon}{\sigma_y}\right)$ is the ratio of the standard deviation of the error term to the standard deviation of y.

Expression 3.1.24 is the generalized expression for expectation of the c^{th} standardized moment.

Specifically, when c = 1, we have

$$E\left(\frac{y-\mu_y}{\sigma_y}\right)^1 = E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right) \quad + R_{\sigma_{\varepsilon,y}}\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right) \quad \right]^1$$

$$\Phi_{y,1} = \rho^1\Phi_{x,1} + R_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,1} \qquad \ldots 3.1.26$$

where

$\Phi_{y,1} = E\left(\frac{y-\mu_y}{\sigma_y}\right)^1$, $\Phi_{x,1} = E\left(\frac{x-\mu_x}{\sigma_x}\right)^1$ and $\Phi_{\varepsilon,1} = E\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)^1$ respectively.

By 3.1.26, we have

$$\rho^1 = \frac{\Phi_{y,1}}{\Phi_{x,1}} = 0 \qquad \ldots 3.1.27$$

This occurs when the error term is well behaved, moreso as the expected value of standardized moment at this point is equal to zero.

Under linear model, c = 1 is the specification corresponding to when $\rho \to 0$. This supports the claims by Rao(1966a) and other co-researchers who ascribed the estimator to situation when there exist poor correlation between the study variables and selection probabilities.

When c = 2,

$$E\left(\frac{y-\mu_y}{\sigma_y}\right)^2 = E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right) \quad + R_{\sigma_{\varepsilon,y}}\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right) \quad \right]^2$$

$$= E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right) \quad \right]^2 + E\left[R_{\sigma_{\varepsilon,y}}\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)\right]^2 + 2\rho^2 R^2_{\sigma_{\varepsilon,y}} E\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)$$

$$\Phi_{y,2} = \rho^2\Phi_{x,2} + R^2_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,2} \qquad \text{...3.1.28}$$

But $\Phi_{x,2} = 1;\ \Phi_{\varepsilon,2} = 1;$ and also $\Phi_{y,2} = 1$.

Therefore,

$$1 = \rho^2 + R^2_{\sigma_{\varepsilon,y}} \qquad \text{…3.1.29}$$

Under linear framework, $R^2_{\sigma_{\varepsilon,y}} < 1$ always. At this point, two scenarios can be identified, namely;

i. when $R^2_{\sigma_{\varepsilon,y}}$ is negligible. Here, $\rho^2 \to 1$.

ii. when $R^2_{\sigma_{\varepsilon,y}}$ is a quantity in [0,1] and $\rho^2 \nrightarrow 1$.

Empirically, we observe that ;

a) if $R^2_{\sigma_{\varepsilon,y}} = 0$ (say), then $\rho^2 = 1$; $\rho = 1$;
b) if $R^2_{\sigma_{\varepsilon,y}} = 0.1$ (say), then $\rho^2 = 0.9$; $\rho = 0.94$;
c) if $R^2_{\sigma_{\varepsilon,y}} = 0.2$ (say), then $\rho^2 = 0.8$; $\rho = 0.89$;
d) if $R^2_{\sigma_{\varepsilon,y}} = 0.3$ (say), then $\rho^2 = 0.7$; $\rho = 0.83$;
e) if $R^2_{\sigma_{\varepsilon,y}} = 0.4$ (say), then $\rho^2 = 0.6$; $\rho = 0.77$;
f) if $R^2_{\sigma_{\varepsilon,y}} = 0.5$ (say), then $\rho^2 = 0.5$; $\rho = 0.71$;
g) if $R^2_{\sigma_{\varepsilon,y}} = 0.6$ (say), then $\rho^2 = 0.4$; $\rho = 0.63$;
h) if $R^2_{\sigma_{\varepsilon,y}} = 0.7$ (say), then $\rho^2 = 0.3$; $\rho = 0.54$;
i) if $R^2_{\sigma_{\varepsilon,y}} = 0.8$ (say), then $\rho^2 = 0.2$; $\rho = 0.45$;
j) if $R^2_{\sigma_{\varepsilon,y}} = 0.9$ (say), then $\rho^2 = 0.1$; $\rho = 0.32$;
k) if $R^2_{\sigma_{\varepsilon,y}} = 1$ (say), then $\rho^2 = 0$; $\rho = 0$

Condition (a) above was assumed by Hansen and Hurwitz(1943) while condition (k) above assumed by Rao(1966a). These extreme conditions are found to be very rare in real life happenings. Suppose we assume conditions (b) to (j) say, we have instances of what we may call “weak”, “moderately low”, “moderately high” and “very high” correlation coefficients. We can conclude here that the appropriateness of the specification parameter, c = 2 is when $R^2_{\sigma_{\varepsilon,y}}$ tends to unity and ρ is “moderately weak”.

When c = 3,

$$E\left(\frac{y-\mu_y}{\sigma_y}\right)^3 = E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right) + R_{\sigma_{\varepsilon,y}}\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)\right]^3$$

$$= E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\right]^3 + 3\rho^2 R_{\sigma_{\varepsilon,y}} E\left(\frac{x-\mu_x}{\sigma_x}\right)^2\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right) + 3\rho R^2_{\sigma_{\varepsilon,y}} E\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)^2 + R^3_{\sigma_{\varepsilon,y}} E\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)^3$$

This implies that

$$\Phi_{y,3} = \rho^3\Phi_{x,3} + 3\rho^2 R_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,1}\Phi_{x,2} + 3\rho R^2_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,2}\Phi_{x,1} + R^3_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,3} \qquad \text{...3.1.30}$$

so that

$$\gamma_y = \rho^3\gamma_x + R^3_{\sigma_{\varepsilon,y}}\gamma_\varepsilon, \qquad \text{...3.1.31}$$

where $\gamma_y = \Phi_{y,3}$; $\gamma_x = \Phi_{x,3}$ and $\gamma_\varepsilon = \Phi_{\varepsilon,3}$ are the skewness coefficients of y, x and ε respectively.

Again, if $R^3_{\sigma_{\varepsilon,y}}$ and hence $R^3_{\sigma_{\varepsilon,y}}\gamma_\varepsilon$ are negligible, then

$$\rho^3 = \frac{\gamma_y}{\gamma_x}, \gamma_x \neq 0. \qquad \text{...3.1.32}$$

Now, $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1 \Rightarrow \gamma_y < \gamma_x$ satisfying $0 < \rho^3 < 1$.

Remark 3.1.3: 3.1.32 above expresses the third power of correlation coefficient as the ratio of the skewness coefficient of y and x.

For c = 4,

$$E\left(\frac{y-\mu_y}{\sigma_y}\right)^4 = E\left[\rho\left(\frac{x-\mu_x}{\sigma_x}\right) \quad + R_{\sigma_{\varepsilon,y}}\left(\frac{\in-\mu_\in}{\sigma_\varepsilon}\right)\quad\right]^4$$

$$= \rho^4 E\left(\frac{x-\mu_x}{\sigma_x}\right)^4 \quad + 6\rho^2 R^2_{\sigma_{\varepsilon,y}} E\left(\frac{x-\mu_x}{\sigma_x}\right)^2 E\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)^2 + R^4_{\sigma_{\varepsilon,y}} E\left(\frac{\varepsilon-\mu_\varepsilon}{\sigma_\varepsilon}\right)^4$$

$$\Phi_{y,4} = \rho^4\Phi_{x,4} + 6\rho^2 R^2_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,2}\Phi_{x,2} + R^4_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,4} \qquad ...3.1.33$$

Now,

$$\Phi_{y,4} - 3 = \rho^4\left[\Phi_{x,4} - 3 + 3\right] + 6\rho^2 R^2_{\sigma_{\varepsilon,y}}\Phi_{\varepsilon,2}\Phi_{x,2} + R^4_{\sigma_{\varepsilon,y}}\ \left[\Phi_{\varepsilon,4} - 3 + 3\right] - 3 \qquad ...3.1.34$$

Now,

$$K_y = \rho^4 K_x + 3\rho^4 + 6\rho^2(1-\rho^2) + (1-\rho^2)^2 K_\varepsilon + 3(1-\rho^2)^2 - 3$$

$$= \rho^4 K_x + 3\rho^4 + 6\rho^2 - 6\rho^4 + (1+\rho^4-2\rho^2)\ \ K_\varepsilon + 3(1+\rho^4-2\rho^2)\ \ - 3$$

$$= \rho^4 K_x + (1-\rho^2)^2 K_\varepsilon \qquad ...3.1.35$$

If $K_\varepsilon = 0$,

$$\rho^4 = \frac{K_y}{K_x};\ K_x \neq 0 \qquad ...3.1.36$$

Expression 3.1.36 represents the forth power of correlation coefficient as the ratio of the kurtosis of the response variable and the kurtosis of the explanatory variable. This can be interpreted as the percentage of kurtosis which is presented by linear model. Certainly, $\rho^4 = \frac{K_y}{K_x} < 1$ is expected when linear relationship is true.

Similarly, under 3.1.23,

$$E(y) = \beta E(x) + E(\varepsilon)$$

$$\Rightarrow \mu_y = \beta\mu_x + \mu_\varepsilon$$

$$= \rho\frac{\sigma_y}{\sigma_x}\mu_x + \mu_\varepsilon\ ;\ \ \mu_\varepsilon = 0,$$

so that

$$\rho^1 = \frac{\mu_y}{\mu_x}\frac{\sigma_x}{\sigma_y} = \frac{CV_x}{CV_y} \qquad \text{...3.1.37}$$

Under ratio estimation, Cochran(1977) have shown that the variance of Ratio estimator is less than SRS when $s_y^2 + R^2 s_x^2 - 2Rs_y s_x = 0$ so that ρ > $\frac{c_x}{2c_y} < 1$ be fulfilled.

Having expressed the correlation coefficient in terms of moments, we can observe the followings:

i) $\rho^1 = \frac{CV_x}{CV_y}$; with $CV_y \neq 0$;

ii) $\rho^2 = 1 - R^2_{\sigma_{\varepsilon,y}}$; $R^2_{\sigma_{\varepsilon,y}}$ does not tend to zero.

iii) $\rho^3 = \frac{\gamma_y}{\gamma_x}$ <1; $\gamma_x \neq 0$ and

iv) $\rho^4 = \frac{K_y}{K_x}$ <1; $K_x \neq 0$.

The question then is how do these moments translate into the specification parameters of the alternative estimators for the target population? Let us define the correlation coefficient as "weak" when 0<ρ≤0.25, "moderately low" when 0.25<ρ≤0.50, "moderately high" when 0.51<ρ≤0.75 and "very high" when 076<ρ<1. We observe certain moments such that when $\rho \to 0$, the specification parameter is c = 1 if and only if $\rho^1 = \frac{CV_x}{CV_y} < 1$. The second moment with c = 2 is the required specification when ρ is "moderately low" especially when $\rho^2 \to 0$ or 0.25<ρ≤0.50, Similarly, when $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1$ and $R^3_{\sigma_{\varepsilon,y}}$ is a small quantity, then c = 3 is the appropriate specification for the proposed estimator and this happens when 0.51<ρ≤0.75 in what is termed here as "moderately high" correlation. When 076<ρ<1, then c =4 would be specified especially as $\rho^4 = \frac{K_y}{K_x} < 1$ and $R^4_{\sigma_{\varepsilon,y}} \to 0$. This occurs when ρ becomes very strong.

3.6 Range of the Specification Parameter c

Considering the generalized transformations defined in 3.1.19 above, we make the following propositions:

Proposition 3.1.5: From 3.1.19 above, let *g(c)* be such a function g:[0,1] for which *f*:[N][0,1][0,1]→[0,1] in the probability measurable space is defined, then *g(c)* = ρ^c, c > 0 satisfies the regularity conditions (i) to (iv) associated with *f* above.

Proof: the result is evidenced in (3.1.27), (3.1.29), (3.1.32) and (3.1.36) above when the stated conditions hold true.

Proposition 3.1.6 Usually in 3.1.19 above, under positive correlation, $0 \leq \rho \leq 1$ and hence $0 \leq \rho^c \leq 1$. However, if $\rho^c > 1$, then the transformation of the form $\rho^{\frac{1}{c}}$ is a necessary substitute for ρ^c satisfying $0 \leq \rho^{\frac{1}{c}} \leq 1$.

Proof: since we desire a transformation of the form *g(c)*: [0,1]→[0,1], then from (3.1.27), (3.1.29) (3.1.32) and (3.1.36) we can see that:

$\rho^c = 0 \Rightarrow \rho^{1/c} = 0^{1/c}$;

$\rho^c = 1 \Rightarrow \rho^{1/c} = 1^{1/c}$.

Thus, ρ^c and $\rho^{1/c}$ are members of the domain g(c):[0,1]→[0,1] hence the proof.

By the propositions above, we can conveniently define the range of the specification parameter in the interval defined by the c^{th} moments as c = [1/4, 4].

3.7 Characterization of estimators in the linear class.

Considering the generalized transformation in 3.1.19 and the range of the specification parameter given above, we can characterize the estimators as follows:

i) $\rho^c=0, \Longrightarrow p_i^* = 1/N$, which is the Rao's(1966a) estimator;
ii) $\rho^c=1, \Longrightarrow p_i^* = p_i$, which is the Hansen and Hurwitz's(1943) estimator;
iii) $0 < \rho^c < 1$, c =1, is the Amahia et al's(1989) estimator;

iv) $0 < \rho^c < 1$, c =1/3, is the Grewal's(1997) estimator;

v) Following our definition of c, we can see that c =[1/4, 4] in which estimators in i to iv above are contained. Thus our proposed estimators in the linear class include the followings;

a) $0 < \rho^c < 1$, c =2 when ρ^2 is moderate ;

b) $0 < \rho^c < 1$, c =3 when $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1$; and

c) $0 < \rho^c < 1$, c =4 when $\rho^4 = \frac{K_y}{K_x} < 1$

When conditions in (a), (b) and (c) above do not hold, then c takes the values ½, 1/3 and ¼ respectively.

Similarly, under negative correlation, our propositions and hence theorems showed that under the transformation, the correlation structure is changed thus, by the symmetric properties of correlation and the derived correlation under the transformation from 1/*x* to *z*, we have the followings:

i. $\rho^c = 0, \Longrightarrow p_i^* = 1/N$ which is the Rao's(1966a) estimator;

ii. $\rho^c = -1, \Longrightarrow p_i^* = p_i$ which is the Hansen and Hurwitz(1943) estimator.

iii. $-1<\rho^c < 0 \Longrightarrow p_i^*$, which is the proposed generalized transformation for use when the study and auxiliary variables are negatively correlated.

CHAPTER FOUR

PROPOSED ALTERNATIVE LINEAR ESTIMATORS IN COMPLEX SURVEYS

4.1 Introduction

In the previous chapter, we developed certain transformations for selection probabilities under positive and negative correlation coefficients between the study and measure of size variables. We showed that the structure of correlation coefficient changes to direct relationship under inverse transformation. We also, provided links between moments in correlation coefficient and statistical properties namely, coefficient of variation, determination, skewness and kurtosis will be established under linear framework.

In this chapter, we shall develop a class of alternative linear estimators which shall be compared with the conventional estimators and also the existing alternative estimators that belong to the linear class using the design based optimality criteria; namely, relative efficiency measured by the relative mean square error (MSE) for PPS sampling design as well as the expected MSE for super-population model that will be derived. An expression for determining the approximate value of the specification parameter c will also be derived so that for a given population, it will provide a necessary guide for the specification of estimators defined by c=1,2,3 and 4 for target populations. The proposed estimators shall be compared with the conventional estimators for a sample sizes of n=2 under PPS design and n = 5 under the Rao-Hartley and Cochran procedure.

4.2 Alternative linear estimators in PPSWR sampling scheme

We consider the homogenous linear estimator (HLE) of the form

$\hat{\tau}_g = \sum_{i\epsilon\Omega} b_{si} I_{si} y_i$, i= 1,2,3, …, N ...4.1

where

$I_{si} = \begin{cases} 0, & if\ i \notin S \\ 1, & if\ i \in S \end{cases}$

and b_{si} are weights not depending on y_i's but on the sample design. Let $\hat{\tau}_{g,c}$ be the estimators of the population total defined by the generalized transformation g under the c^{th} moments, then under PPSWR sampling, $b_{si} = 1/(np_{i,g}^*)$ so that our estimator of population total becomes

$$\hat{\tau}_{g,c} = \frac{1}{n}\sum_{i\epsilon\Omega}\frac{y_i}{p_{i,g}^*}, \qquad ...4.2$$

where

$$p_{i,g}^* = \frac{1-g(c)}{N} + g(c)p_i, \text{ with } g(c) = \rho^c, c = 1,2,3,4; 0<\rho<1, c>0 \qquad ...4.3$$

so that 4.2 is the proposed estimators realized by propositions 3.2, 3.3 and theorem 3.1 to 3.4 above.

Our interest in this study is therefore, to develop linear estimators of population totals in PPS sampling scheme defined by the moments, c = 1, 2, 3 and 4 only.

4.2.1 Design based bias of the proposed estimators

Now, our proposed estimator under PPSWR sampling scheme is bias as

$$E_p(\hat{\tau}_g) = \frac{1}{n}\sum_{i\epsilon\Omega}\frac{I_{si}E(y_i)}{p_{i,g}^*} = \frac{1}{n}\sum_{i\epsilon s}\frac{y_i p_i}{p_{i,g}^*} \neq Y$$

Theorem 4.1: Let y_i, {i=1,2,3, …, N} be a finite population under study and let x_i, (i=1,2,3, …, N} be the values of the auxiliary variable associated with the i^{th} study variable yielding the coordinates, (x_i, y_i). Suppose that these variables are correlated such that $0<\rho<1$, then the variance of the estimator of population total is defined by

$$B_p(\hat{\tau}_{g,c}) = (1-\rho^c)N^2\text{COV}\left(\frac{Np_i}{1-\rho^c+N\rho^c p_i}, p_i\right) \qquad ...4.4$$

Proof: The design based bias denoted as B_p is therefore,

$$B_p(\hat{\tau}_{g,c}) = E_p(\hat{\tau}_{g,c}) - Y$$

$$= \sum_{i\epsilon\Omega} \frac{I_{si}E(y_i)}{p^*_{i,g}} - Y$$

$$= \sum_{i\epsilon\Omega} \left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right) y_i$$

$$= \sum_{i\epsilon\Omega} \left(\frac{Np_i}{1-\rho^c+N\rho^c p_i} - 1\right) y_i$$

$$= (1-\rho^c)N^2\text{COV}\left(\frac{Np_i}{1-\rho^c+N\rho^c p_i}, p_i\right) > 0 \qquad \ldots 4.5$$

Now, let $k = N^2\text{COV}\left(\frac{Np_i}{1-\rho^c+N\rho^c p_i}, p_i\right)$, then from 4.5 above, the bias of the proposed estimator is a decreasing function as ρ→1.

Importantly, $B_p(\hat{\tau}_{g,1}) < B_p(\hat{\tau}_{g,2}) < B_p(\hat{\tau}_{g,3}) < B_p(\hat{\tau}_{g,4})$ for all values of c. The term $\text{COV}\left(\frac{Np_i}{1-\rho^c+N\rho^c p_i}, p_i\right)$ is the covariance between $\frac{Np_i}{1-\rho^c+N\rho^c p_i}$ and p_i.

4.2.2 Design based variance of the proposed estimators

Theorem 4.2.: Let y_i, {i=1,2,3, …, N} be a finite population under study and let x_i, (i=1,2,3, …, N} be the values of the auxiliary variable associated with the i^{th} study variable yielding the coordinates, (x_i,y_i). Suppose that these variables are correlated such that 0<ρ<1, then the variance of the estimator of population total is defined by

$$V_p(\hat{\tau}_{g,c}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega} \frac{I_{si}y_i^2 p_i}{p^{*2}_{i,g}} - \left(\sum_{i\epsilon\Omega} \frac{I_i y_i p_i}{p^*_{i,g}} -\right)^2\right] \qquad \ldots 4.6$$

Proof:

$$V_p(\hat{\tau}_{g,c}) = Var\left(\frac{1}{n}\sum_{i\epsilon s} \frac{y_i}{p^*_{i,g}}\right)$$

$$= \frac{1}{n^2} Var\left(\sum_{i\epsilon s} \frac{y_i}{p^*_{i,g}}\right)$$

$$= \frac{1}{n} Var(z); \quad z = \frac{y_i}{p_{i,g}^*}$$

Since Var(z) = E(z^2)-E^2(z), it follows that

$$V_p(\hat{\tau}_{g,c}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega} \frac{I_{si} y_i^2 p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega} \frac{I_i y_i p_i}{p_{i,g}^*}\right)^2\right] \quad \text{hence the proof.}$$

We noted that $\hat{\tau}_{g,c}$ is already biased as such, it will be sufficient to consider the mean square error (MSE) for our inference.

4.2.3 Mean squared error of the proposed estimators

Following the classical definition of MSE in (1.13) above, we define the MSE of the proposed estimators as

$$MSE(\hat{\tau}_{g,c}) = V_p(\hat{\tau}_{g,c}) + B^2(\hat{\tau}_{g,c})$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega} \frac{I_{si} y_i^2 p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega} \frac{I_i y_i p_i}{p_i^*}\right)^2\right] + \left[\left(\sum_{i\epsilon\Omega} \frac{I_i y_i p_i}{p_i^*} - Y\right)^2\right] \quad \ldots 4.7$$

4.2.4 Expected MSE of the proposed Estimators.

Here, we consider the super-population model defined by

$$y = \beta p_i + \varepsilon \quad \ldots 4.8$$

With $E(\varepsilon/p_i) = 0$, $Cov(\varepsilon_i \varepsilon_j) = 0$ and $E(\varepsilon_i^2) = a p_i^g$

Theorem 4.3: Under super-population model, the expected MSE of the proposed PPS estimators involving multiple characteristics is

$$\xi MSE(\hat{\tau}_{g,c}) = \frac{a}{n}\left[\sum_{i\epsilon\Omega} \frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}}\right] + \frac{\beta^2}{n}\left[\sum_{i\epsilon\Omega} \frac{p_i^3}{p_{i,g}^{*2}} - (\sum_{i=1}^{N} \frac{p_i^2}{p_{i,g}^*})^2\right] +$$

$$\left[\beta \sum_{i\epsilon\Omega} \left(\frac{I_{si} p_i}{p_{i,g}^*} - 1\right) p_i\right]^2 \quad \ldots 4.9$$

Proof: The result is obvious when we take the model based expectation over the MSE of the design based estimator. Thus,

$$\xi MSE\left(\hat{\tau}_{g,c}\right) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega}\frac{I_i\xi(y_i)p_i}{p_{i,g}^{*}}\right)^2\right] + \left[\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)\xi(y_i)\right]^2$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \sum_{i=1}^{N}\frac{\xi(y_i^2)p_i^2}{p_{i,g}^{*2}} - \sum\sum_{i=1}^{N}\frac{\xi(y_iy_j)p_ip_j}{p_{i,g}^{*}p_{j,g}^{*}}\right] +$$

$$\left[\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)\xi(y_i)\right]^2$$

$$= \frac{a}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}}\right] + \frac{\beta^2}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^3}{p_{i,g}^{*2}} - (\sum_{i=1}^{N}\frac{p_i^2}{p_{i,g}^{*}})^2\right] + \left[\beta\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)p_i\right]^2$$

Under super-population model, the expected per unit bias in terms of β is very negligible as such, inference based on the expected variance would be sufficient. We now show that the anticipated bias is negligible.

Considering (4.9) above, Let the bias be

$$\xi B(\hat{\tau}_{g,c}) = \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)\xi(y_i)$$

$$= \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)\xi(\beta p_i + \varepsilon_i)$$

$$= \beta\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)p_i,$$

then we have the following theorem.

Theorem 4.4: Under super-population model, the expected bias is very negligible as $\xi B^2(\hat{\tau}_{g,c}) = \nabla \to 0$

Proof: Considering the anticipated bias from the model,

$$\xi B(\hat{\tau}_{g,c}) = \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)\xi(y_i)$$

$$=\beta \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)p_i$$

when $\frac{p_i}{p_i^*} = 1$ then $\beta \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)p_i = 0$ and $B(\hat{\tau}_{g,c}) = 0$

when $\frac{p_i}{p_i^*} < 1$ then $\beta \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)p_i < 1$ and hence, $B(\hat{\tau}_{g,c}) \to \nabla < 1$ especially when $\beta \to 1$.

Also, when $\frac{p_i}{p_i^*} > 1$ then $\beta \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)p_i > 0$ and $\xi B(\hat{\tau}_{g,c}) \to \nabla < 1$ especially when $\beta \to 1$. Since $\xi B(\hat{\tau}_{g,c}) = 0$ when $\frac{p_i}{p_i^*} = 1$ which is a necessary condition for unbiasness, we can conveniently state that in the case of a biased estimator the condition becomes $0 < \xi B(\hat{\tau}_{g,c}) < 1$.

Alternatively, by Cauchy-Schwarz inequalities,

$$\xi B^2(\hat{\tau}_{g,c}) = \left[\beta \sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)p_i\right]^2 = \beta^2\left[\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p^*_{i,g}} - 1\right)p_i\right]^2$$

$$\leq \beta^2\left[\sum_{i\in\Omega}\left(\frac{p_i}{p_i^*} - 1\right)^2 \sum_{i\in\Omega} p_i^2\right]$$

But $\sum_{i\in\Omega}\left(\frac{p_i}{p_i^*} - 1\right)^2 \sum_{i\in\Omega} p_i^2 = \nabla \ll 1$

so that $0 < \left[\sum_{i\in\Omega}\left(\frac{p_i}{p_i^*} - 1\right)^2 \sum_{i\in\Omega} p_i^2\right] < 1$ always.

Therefore, $\xi B^2(\hat{\tau}_{g,c}) = \nabla \to 0$.

Thus, under super-population model, the expected bias per unit is very negligible especially when $0 < \beta \leq 1$ as such, inference based on the expected variance would be sufficient.

4.2.5 Comparison of the proposed estimators under super-population model.

From the optimality criteria, we know that an estimators, $\hat{\tau}_{g,c}$ (say) is relatively more efficient than another estimator $\hat{\tau}_c$ (say), when $V_p(\hat{\tau}_{g,c}) \leq V_p(\hat{\tau}_c)$ in terms of sampling design or $\xi V_p(\hat{\tau}_{g,c}) \leq \xi V_p(\hat{\tau}_c)$ in terms of super-population model.

Now,

$$V(\hat{\tau}_{HH}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}y_i^2}{p_i} - (\textstyle\sum_{i=1}^{N} y_i)^2\right] \qquad \ldots 4.10$$

Under super-population model,

$$\xi V_p(\hat{\tau}_{HH}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)}{p_i} - (\textstyle\sum_{i=1}^{N} \xi(y_i))^2\right]$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)}{p_i} - \left(\textstyle\sum_{i=1}^{N} \xi(y_i^2) + \sum_{i\neq j=1}^{N}\sum \xi(y_i y_j)\right)\right]$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{[\beta^2 p_i^2 + a p_i^g]}{p_i} - \textstyle\sum_{1=1}^{N}[\beta^2 p_i^2 + a p_i^g] - \sum\sum_{1=1}^{N}\beta^2 p_i p_j\right]$$

$$= \frac{a}{n}\left[\textstyle\sum_{i\epsilon\Omega} p_i^{g-1} - \sum_{i=1}^{N} p_i^g\right] + \frac{\beta^2}{n}\left[\textstyle\sum_{i\epsilon\Omega} p_i^2 - (\sum_{i=1}^{N} p_i)^2\right]$$

$$= \frac{a}{n}\left[\textstyle\sum_{i\epsilon\Omega} p_i^{g-1} - \sum_{i=1}^{N} p_i^g\right] + 0 \qquad \ldots 4.11$$

The expected variance of the proposed estimator is

$$\xi V\ (\hat{\tau}_{g,c}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega}\frac{I_i\xi(y_i)p_i}{p_{i,g}^{*}}\right)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \sum_{i=1}^{N}\frac{\xi(y_i^2)p_i^2}{p_{i,g}^{*2}} - \sum\sum_{i=1}^{N}\frac{\xi(y_i y_j)p_i p_j}{p_{i,g}^{*}p_{j,g}^{*}}\right]$$

$$= \frac{a}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}}\right] + \frac{\beta^2}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^3}{p_{i,g}^{*2}} - (\textstyle\sum_{i=1}^{N}\frac{p_i^2}{p_{i,g}^{*}})^2\right] \qquad \ldots 4.12$$

Now, let $n\left[\xi V_p\left(\hat{\tau}_{g,HH}\right) - \xi V_p\left(\hat{\tau}_{HH}\right) = n\nabla\right]$ so that

$$n\nabla = \mathrm{a}\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}} + \beta^2\sum_{i\epsilon\Omega}\frac{p_i^3}{p_{i,g}^{*2}} - \left(\sum_{i=1}^{N}\frac{p_i^2}{p_{i,g}^*}\right)^2 - \left(\sum_{i\epsilon\Omega}p_i^{g-1} - \sum_{i=1}^{N}p_i^g\right)$$

$$= \mathrm{a}\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}} - \sum_{i\epsilon\Omega}p_i^g\left(\frac{1-p_i}{p_i}\right) + \beta^2\left[\sum_{i\epsilon\Omega}\frac{p_i^3}{p_{i,g}^{*2}} - \left(\sum_{i=1}^{N}\frac{p_i^2}{p_i^*}\right)^2\right]$$

$$= \mathrm{a}\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}} - \sum_{i\epsilon\Omega}p_i^g\left(\frac{1-p_i}{p_i}\right) + \beta^2 V\left(\frac{p_i}{p_{i,g}^*}\right) \quad \ldots 4.13$$

Now, let $\delta = \sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}} - \sum_{i\epsilon\Omega}p_i^g\left(\frac{1-p_i}{p_i}\right)$

$$= \mathrm{a}\sum_{i\epsilon\Omega}\frac{p_i^{g-1}(1-p_i)}{p_{i,g}^{*2}}\left(p_i^2 - p_{i,g}^{*2}\right) \quad \ldots 4.14$$

Satisfying

$$n\nabla = a\delta + \beta^2 D \quad \ldots 4.15$$

empirically,

when ρ=0,

$$D = \mathrm{Var}(p_i/p_{i,g}^*) > 0 \quad \ldots 4.16$$

and

as $\rho > 0$, $D = \mathrm{Var}(p_i/p_{i,g}^*) \to 0$ …4.17

Since in most real life scenario, $\rho \neq 0$ always, we consider (4.17) above as most ideal for surveys and hence, inference based on $a\delta$ will be sufficient.

Now, let

$$\delta = \sum_{i=1}^{n} b_i^* c_i^* = \sum_{i\epsilon\Omega}\frac{p_i^{g-1}(1-p_i)}{p_{i,g}^{*2}}\left(p_i^2 - p_{i,g}^{*2}\right) \quad \ldots 4.18$$

where

$$b_i^* \quad = \frac{p_i^{g-1}(1-p_i)}{p_{i,g}^{*2}} \qquad \ldots 4.19$$

and

$$c_i^* \; = (p_i^2 - p_{i,g}^{*2}) \qquad \ldots 4.20$$

Then, we can as well observe that

$\sum_{I=1}^{n} c_i^* < 0, \; if \; 0 < \rho < 1,$

or

$\sum_{i=1}^{n} c_i^* = 0, \text{if}\, \rho = 1.$

We now state the appropriate Lemma useful in this study.

Lemma 4.2.1: Let $0 \leq b_1 \leq b_2 \leq \ldots \leq b_n$ and $c_1 \leq c_2 \leq \ldots \leq c_n$ satisfy $\sum_{i=1}^{n} c_i^* \geq 0,$

then $\sum_{i=1}^{n} b_i c_i^* \geq 0$

Proof : Due to Royall(1970).

Let k denote the greatest integer i for which $c_i \leq 0$, then

$$\sum_{i=1}^{n} b_i c_i^* = \sum_{i=1}^{k} b_i c_i \; + \sum_{i=k+1}^{n} b_i c_i^*$$

$$\geq b_k \sum_{i=1}^{k} c_i \; + b_{k+1} \sum_{i=k+1}^{n} c_i$$

$$= b_k \left(\sum_{i=1}^{k} c_i \; + \sum_{i=k+1}^{n} c_i \; \right) + b_{k+1} \sum_{i=k+1}^{n} c_i$$

$$= b_k \sum_{i=1}^{n} c_i \; + (b_k - b_{k+1}) \sum_{i=k+1}^{n} c_i$$

$$\geq 0$$

Lemma 4.2.2: Let p_i be positive with $\sum_{i=1}^{N} p_i = 1$ and let $p_{i,g}^{*} = \frac{1-\rho^c}{N} + \rho^c p_i$, *0<ρ<1, 1≤i≤N,* then

$$V\left(\frac{1}{p_{i,g,c}^{*}}\right) \leq V\left(\frac{1}{p_{i,g,c'}^{*}}\right)$$

where $p^*_{i,g}$ and $p^*_{i,g'}$ are the selection probabilities determined by c and c' moments when $c \neq c'$.

Proof:

Let $p_{i,g,c}^{*} = \frac{1-\rho^c}{N} + \rho^c p_i$, $0<\rho<1$, $c \geq 1$ and $p_{i,g,c'}^{*} = \frac{1-\rho^{c'}}{N} + \rho^{c'} p_i$, ρ=0, then

If (i) $N \geq 1/p_i$, then $1/p_i \leq 1/p_{i,g}^* \leq 1/N$

(ii) $N < 1/p_i$, then $1/p_i > 1/p_{i,g}^* > 1/N$

Hence, $\left|\frac{1}{p_i} - N\right| \geq \left|\frac{1}{p_{i,g}^{*}} - N\right| \ \forall i$ …4.21

Therefore, $V\left(\frac{1}{p_i}\right) = \sum_{i=1}^{N} p_i (\frac{1}{p_i} - N)^2$

$$\geq \sum_{i=1}^{N} p_i (\frac{1}{p_{i,g}^{*}} - N)^2$$

$$= V\left(\frac{1}{p_{i,g}^{*}}\right)$$ …4.22

Lemma 4.3.3: Let p_i be positive with $\sum_{i=1}^{N} p_i = 1$ and let $p_{i,g}^{*} = \frac{1-\rho^c}{N} + \rho^c p_i$, $0<\rho<1$, $1 \leq i \leq N$ and $\sum_{i=1}^{N} p_{i,g}^{*} = 1$, then

$V\left(\frac{1}{p_i^{*}}\right) \leq V\left(\frac{1}{p_i}\right)$ where $p_{i,g}^*$ and p_i, are the selection probabilities, and hence, weights of each selected unit due to the proposed estimator and that of Hansen and Hurwitz.

Proof:

Let $p^{*}_{i,g} = \frac{1-\rho^c}{N} + \rho^c p_i$, 0<ρ<1, c ≥ 1 and $p'_{i,g} = \frac{1-\rho^c}{N} + \rho^c p_i$, ρ=1, then

$$V\left(\frac{1}{p^{*}_{i,g}}\right) < V\left(\frac{1}{p_{\ i}}\right) \qquad \ldots 4.23$$

Corollary: Under super-population model specified in 4.8 above, it is clear that $\hat{\tau}_{g,c}$ has smaller variance than $\hat{\tau}_{HH}$.

Lemma 4.2.4: Let p_i be positive with $\sum_{i=1}^{N} p_i = 1$ and let $p^{*}_{i,g} = \frac{1-\rho^c}{N} + \rho^c p_i$, 0<ρ<1, 1≤i≤N $\sum_{i=1}^{N} p^{*}_{i,g} = 1$ and also $p'_{i,g} = \frac{1-\rho^{c'}}{N} + \rho^{c'} p_i$, 0<ρ<1, c'>0, c>0 and $\sum_{i=1}^{N} p'_{i,g} = 1$

then

$$V\left(\frac{1}{p^{*}_{i,g}}\right) \leq V\left(\frac{1}{p_{i,}}\right)$$

where $p_{i,g}$* and $p_{i,}$ are the selection probabilities, and hence, weights of each selected unit due to the proposed estimator and that of Hansen and Hurwitz's(1943) estimator.

Proof:

If $p^{*}_{i,g} = \frac{1-\rho^c}{N} + \rho^c p_i$, 0<ρ<1, c ≥ 1 and $p'_{i,g} = \frac{1-\rho^{c'}}{N} + \rho^{c'} p_i$, ρ=1,

then

$$V\left(\frac{1}{p^{*}_{i,g}}\right) < V\left(\frac{1}{p'_{i,g}}\right) \qquad \ldots 4.24$$

$$V\left(\frac{1}{p^{*}_{i,g}}\right) = \rho^c V\left(\frac{1}{p_i}\right)$$

and

$$V\left(\frac{1}{p'_{i,g}}\right) = \rho^{c'} V\left(\frac{1}{p_i}\right)$$

So that 4.24 occurs when

$$\left(\rho^{c} - \rho^{c'}\right) \leq 0 \qquad \ldots 4.25$$

which is applicable in the generalized class of alternative estimators in PPSWR sampling.

Now, If we assume that $\rho^{c'} = 1$ and let c >0 when 0 < ρ < 1, then we can compare the estimators by using the condition in 4.24 above as follows;

a) when c =1 and 0 < ρ < 1, then $(\rho^{1} - 1) \leq 0$. Here, the estimator by Amahia et al(1989) fares better than the conventional estimator;

b) when c >1 and 0 < ρ < 1, then $(\rho^{c} - 1) \leq 0$. Again, the proposed estimators with c>1 are more efficient than the conventional estimators. Similarly, when 0<c<1 and 0 < ρ < 1, then $(\rho^{c} - 1) \leq 0$.

c) When c =1/3 and 0 < ρ < 1, then $\left(\rho^{1} - \rho^{1/3}\right) \leq 0$, hence, the estimator by Amahia et al(1989) fares better than Grewal's(1997) estimator;

d) When c>1 and 0 < ρ < 1, then $\left(\rho^{c} - \rho^{1/3}\right) \leq 0$. The proposed class of linear estimators with c>1 are relatively more efficient than the Grewal's(1997) estimator as $\rho \to 1$

4.2.6 Determination of Approximate value of *c.*

Studies have shown that the value of g useful in estimation ranges between 0 and 2 inclusive. Amahia *et al*(1989) have shown that the value of g is given by

$$g > \frac{2\rho p_i}{p_i^*} + \frac{1}{1-p_i} - \frac{(1+\rho)p_i}{p_i^* + p_i} \qquad \ldots 4.26$$

Theorem 4.5: Under super-population model, the approximate value of the specification parameter, c that minimizes the ξMSE is given by

$$c \cong \left|\frac{\log(\eta)}{\log(\rho)}\right| \rho \neq 0 \text{ or } 1, c > 0 \qquad \ldots 4.27$$

Proof: From 4.19 above, we defined $b_i^* = \frac{p_i^{g-1}(1-p_i)}{p_{i,g}^{*2}}$

So that

$$\frac{db_i^*}{dp_i} = \frac{p_i^{*2}\frac{d}{dp_i}\left[(1-p_i)p_i^{g-1}\right] - (1-p_i)p_i^{g-1}\frac{d}{dp_i}p_i^{*2}}{p_i^{*4}} = 0$$

$$\Rightarrow p_i^{*2}\left[(g-1)p_i^{g-2} - gp_i^{g-1}\right] - 2\rho^c p_i^* p_i^{g-1}(1-p_i) = 0$$

$$p_i^*\left[(g-1)p_i^{g-2} - gp_i^{g-1}\right] - 2\rho^c p_i^{g-1}(1-p_i) = 0$$

$$\left[\left[\frac{1-\rho^c}{N} + \rho^c p_i\right](g-1)p_i^{g-2} - gp_i^{g-1}\right] - 2\rho^c p_i^{g-1}(1-p_i) = 0$$

$$\Rightarrow Ap_i^{g-2} - A\rho^c p_i^{g-2} - Bp_i^{g-1} + B\rho^c p_i^{g-1} + AN\rho^c p_i^{g-1} - g\rho^c p_i^{g} - 2\rho^c p_i^{g-1}$$

$$+2\rho^c p_i^g = 0$$

$$Ap_i^{g-2} - Bp_i^{g-1} = \left[Ap_i^{g-2} - Bp_i^{g-1} - ANp_i^{g-1} + gp_i^g + 2p_i^{g-1} - 2p_i^g\right]\rho^c$$

$$\Rightarrow \Phi_{1,p,g} = \Phi_{2,p,g}\rho^c$$

$$\rho^c = \left|\frac{\Phi_{1,p,g}}{\Phi_{2,p,g}}\right| = \eta$$

Therefore,

$$c \cong \left|\frac{\log(\eta)}{\log(\rho)}\right| \quad \rho \neq 0 \text{ or } 1, c > 0,$$

where

$A = \frac{g-1}{N}$,
$B = \frac{g}{N}$,

$$\Phi_{1,p,g} = Ap_i^{g-2} - Bp_i^{g-1}$$

and

$$\Phi_{2,p,g} = Ap_i^{g-2} - Bp_i^{g-1} - ANp_i^{g-1} + gp_i^g + 2p_i^{g-1} - 2p_i^g \ .$$

Here, there are N values of c thereby giving us a range of values determined by p_i. The choice of c is therefore determined by Min p_i and Max p_i giving us the interval containing the true c that defined the developed estimators.

Remark 1: The value of c depends on g, ρ and N.

Remark 2: just like g, c occurs in a convex region . Empirical evidence shows that as N becomes large and $\rho < 1$, $c < 5$ provide a non-uniform selection probabilities for estimation.

4.3 Alternative linear estimator under negative correlation.

In this section, we utilized the modified Hansen and Hurwitz estimator in order to estimate population characteristics namely, population totals, bias and variances under negative correlation.

We have earlier stated that when $y \propto 1/x$, then under a transformation, y = kz which translates the correlation from negative to positive. In this case, we can define the Modified Hansen and Hurwitz's Estimator (MHHE) as:

$$\hat{\tau}_{g,c} = \frac{1}{n}\sum_{i\epsilon\Omega}\frac{y_i}{p^*_{i,g}} ,$$

where

$$p^*_{i,g} = \frac{1-g(c)}{N} + g(c)p_i, \text{ with } g(c) = \rho^c, \ c = 1,2,3,4$$

$0<\rho<1$, $c>0$, $p_i = z_i/Z$ as defined by theorems 3.4 above.

4.3.1 Bias of the proposed estimator under negative correlation.

The bias is therefore,

$$B_p\left(\hat{\tau}_{g,c}\right) = E_p\left(\hat{\tau}_{g,c}\right) - Y$$

$$= (1-\rho^c)N^2 COV\left(\frac{Np_i}{1-\rho^c+N\rho^c p_i}, p_i\right) \qquad \text{…4.28}$$

4.3.2 Design Based Variance of the proposed estimator under negative correlation.

Theorem 4.6: Let y_i, {i=1,2,3, …, N} be a finite population under study and let x_i, (i=1,2,3, …, N} be the values of the auxiliary variable associated with the i^{th} study variable yielding the coordinates, (x_i,y_i). suppose that these variables are correlated such that $0<\rho<1$, then the variance of the estimator of population total is defined by

$$V_p\left(\hat{\tau}_g\right) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}y_i^2 p_i}{p_{i,g}^{*2}} - \sum_{i\epsilon\Omega}\left(\frac{I_i y_i p_i}{p_{i,g}^{*}}\right)^2\right] \qquad \ldots 4.29$$

Proof:

$$V_p\left(\hat{\tau}_{g,N}\right) = Var(\frac{1}{n}\sum_{i\epsilon s}\frac{y_i}{p_{i,g}^{*}})$$

$$= \frac{1}{n^2}Var(\sum_{i\epsilon s}\frac{y_i}{p_{i,g}^{*}})$$

$$= \frac{1}{n}Var(z); \quad z = \frac{y_i}{p_{i,g}^{*}}$$

Since Var(z) = $E(z^2)-E^2(z)$,

it follows that

$$V_p\left(\hat{\tau}_g\right) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}y_i^2 p_i}{p_{i,g,N}^{*2}} - \left(\sum\frac{I_i y_i p_i}{p_{i,g,N}^{*}}\right)^2\right] \quad \text{hence the proof.}$$

We note that the estimator is already biased as such, it will be sufficient to consider the mean square error (MSE) for our inference.

4.3.3 Mean Squared error of the proposed Estimator

Theorem 4.7: Following the classical definition of MSE of an estimator, the MSE of the proposed estimator as

$$MSE(\hat{\tau}_{g,c}) = V_p(\hat{\tau}_{g,c}) + B^2(\hat{\tau}_{g,c})$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}y_i^2p_i}{p_{i,g,N}^{*2}} - \left(\sum\frac{I_iy_ip_i}{p_{i,g,N}^{*}}\right)^2\right] + \left[\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g,N'}^{*}} - 1\right)y_i\right]^2 \quad \ldots 4.30$$

Proof: It follows from the definition of MSE and by substituting expression for variances and bias derived.

4.34 Model Based Variance of PPS estimator under negative correlation.

Again, we consider the super-population model defined by

$$y = \beta p_i + \varepsilon \quad \ldots 4.31$$

With $E(\varepsilon/p_i) = 0,\ Cov(\varepsilon_i\varepsilon_j) = 0\ and\ E(\varepsilon_i^2) = ap_i^g$

Theorem 4.8: Under the super-population model, the expected variance is given by

$$\xi V_\epsilon(\hat{\tau}_g) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}y_i^2p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega}\frac{I_iy_ip_i}{p_{i,g}^{*}}\right)^2\right] + \left[\sum_{i\epsilon\Omega}\left(\frac{I_{si}p_i}{p_{i,g}^{*}} - 1\right)y_i\right]^2 \quad \ldots 4.32$$

The proof is similar with that in theorem 4.3 above.

4.35 Comparison of the proposed with conventional estimators under negative correlation

From the optimality criteria, we know that an τ^* (say) is relatively more efficient than another estimator τ (say), when $V_p(\tau^*) \leq V_p(\tau)$ in terms of sampling design or $\xi V_p(\tau^*) \leq \xi\, V_p(\tau)$ in terms of super-population model.

Now,

$$V(\hat{\tau}_{HH}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}y_i^2}{p_i} - (\textstyle\sum_{i=1}^{N} y_i)^2\right] \quad \ldots 4.33$$

Under super-population model,

$$\xi V_p(\hat{\tau}_{HH}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)}{p_i} - (\sum_{i=1}^{N}\xi(y_i))^2\right]$$

$$= \frac{a}{n}\left[\sum_{i\epsilon\Omega}p_i^{g-1} - \sum_{i=1}^{N}p_i^{g}\right] + 0 \qquad \ldots 4.34$$

The expected variance of the proposed estimator is

$$\xi V_p(\hat{\tau}_{g,HH}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega}\frac{I_i\xi(y_i)p_i}{p_{i,g}^{*}}\right)^2\right]$$

$$= \frac{a}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}}\right] + \frac{\beta^2}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^3}{p_{i,g}^{*2}} - (\sum_{i=1}^{N}\frac{p_i^2}{p_{i,g}^{*}})^2\right] \qquad \ldots 4.35$$

4.36 Mean squared error of the proposed Modified Hansen and Hurwitz Estimator.

The expected mean square error of the modified estimator estimators is given as

$$\xi V_p(\hat{\tau}_{g,c}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{I_{si}\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega}\frac{I_i\xi(y_i)p_i}{p_{i,g}^{*}}\right)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega}\frac{\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \sum_{i=1}^{N}\frac{\xi(y_i^2)p_i^2}{p_{i,g}^{*2}} - \sum\sum_{i=1}^{N}\frac{\xi(y_iy_j)p_ip_j}{p_{i,g}^{*}p_{j,g}^{*}}\right]$$

$$= \frac{a}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}}\right] + \frac{\beta^2}{n}\left[\sum_{i\epsilon\Omega}\frac{p_i^3}{p_{i,g}^{*2}} - (\sum_{i=1}^{N}\frac{p_i^2}{p_{i,g}^{*}})^2\right] \qquad \ldots 4.35$$

By this modification, inferential procedures are similar to that used for Hansen and Hurwitz estimator by utilizing the new correlation coefficient given as

$\rho = \rho_{y,z}$ instead of $\rho = \rho_{y,x}$,

where

$$\rho_{y,z} = \frac{cov(y,z)}{\sigma_y\sigma_z}$$

Under the super-population model, Bansal and Singh(1995) have shown that the value of β^2 that minimizes the expected MSE is given by

$$\beta^2 = \frac{\rho^2}{1-\rho^2}\frac{a\sum_{i\epsilon\Omega} p_i^g}{N\sigma_p^2} \qquad \ldots 4.36$$

where

$$\sigma_p^2 = \frac{1}{N}\left[\sum_{i\in\Omega} p_i^2 - \frac{(\sum_{i\in\Omega} p_i)^2}{N}\right] \qquad \ldots 4.37$$

4.4 Alternative linear estimators in πPS sampling design.

4.4.1 Introduction.

In the previous section, a class of alternative linear estimators were proposed for use under PPSWR sampling design for cases of both positive and negative correlation as they relate with coefficients of variation, determination, skewness and kurtosis. In this section, we consider also utilize the estimators proposed in PPSWR sampling, and modify them for use with PPSWOR sampling design otherwise, called πPS sampling design.

This scheme was proposed by Horvitz and Thompson(1952) and is popularly called Horvitz-Thompson Estimator (HTE), otherwise called πPS sampling by Hanurav(1967). Under this scheme, we shall first consider the HTE along with some alternative estimators due to Rao, Amahia, Grewal, Ekaette which belong to the scheme for which the sample size, n=2 and the random Group, especially the Rao-Hartley-Cochran Procedure with respect to our proposed estimator.

Definition 4.4.1:

Let $\Omega \equiv \{u_1, u_2, \ldots, u_N\}$ be a finite population of N identifiable units and $Y \equiv \{y_1, y_2, \ldots, y_N\}$ be a vector of values of $y_i = y(u_i)$, the value assumed by u_i by a real valued variate Y. Let X be a positive valued variate, $\{x_1, x_2, \ldots, x_N\}$ presumed to be correlated with y and x_i being the value of X assumed on u_i, (i=1,2,3, …, N). We define $p_i = x_i/X$; $X=\Sigma x_i$ where p_i is the normed-size measure.

Let us assume a sampling scheme for which the inclusion probability is $\pi_i = np_i$. We shall assume here that the selection procedure is Draw-by-Draw due to *H*orvitz and Thompson for which the generalized first order inclusion probability in which $\sum_{i=1}^{N} \pi_i = 2$ is

$$\pi_{i=} \frac{p_i}{d}\left[\frac{(1-ap_i)(1-2p_i)}{(1-p_i)(1-2bp_i)} + \sum_{i=1}^{N} \frac{p_j(1-ap_j)}{(1-p_j)(1-2bp_j)}\right] \qquad \ldots 4.38$$

For a=0, b=0 and d=1, the expression in 4.38 becomes

$$= \frac{p_i}{1}\left[\frac{1-p_i-p_i}{(1-p_i)} + \sum_{i=1}^{N} \frac{p_j}{(1-p_j)}\right]$$

$$= p_i\left[1 + \sum_{i=1}^{N} \frac{p_j}{(1-p_j)} - \frac{p_i}{(1-p_i)}\right] \qquad \ldots 4.39$$

This selection procedure supposes that the first unit is selected with probability p_i while the second unit, p_j, is selected with probability 1/(1-p_i) if p_i is first selected. Similarly, if p_j is first selected, then, the pi would be selected with probability 1/(1-p_i).

The Joint inclusion probability π_{ij} is the conditional probability of selecting unit j given that unit i has been selected and vice versa. Thus

$$\pi_{ij} = p_i p_{j/i} + p_j p_{i/j},$$

so that

$$\pi_{ij} = \frac{p_i p_j}{d}\left[\frac{(1-ap_i)}{(1-p_i)(1-2bp_i)} + \frac{(1-ap_j)}{(1-p_j)(1-2bp_j)}\right]$$

$$= p_i p_j\left[\frac{1}{(1-p_i)} + \frac{1}{(1-p_j)}\right] \qquad \ldots 4.40$$

Then the HT estimator of population total is

$$\hat{\tau}_g = \sum_{i\epsilon\Omega} b_{si}\, I_{si}\, y_i$$

with I_{si} as earlier defined. The changing factor here is the weight due to the inclusion probabilities, π_i's. Thus,

$$\hat{\tau}_{HT} = \sum_{i\epsilon s} \frac{y_i}{\pi_i} \qquad \ldots 4.41$$

where

$$\pi_i = np_i = \frac{nx_i}{X},\; X = \sum_{i=1}^{N} x_i, \qquad \ldots 4.42$$

and its variance is given by

$$V(\hat{\tau}_{HT}) = \sum_{i=1}^{N}\left(\frac{1}{\pi_i} - 1\right) y_i^2 + \sum_{i<j}^{N}\sum\left(\frac{\pi_{ij}}{\pi_i\pi_j} - 1\right) y_i y_j \qquad \ldots 4.43$$

When the study variable y and the measure of size x are poorly correlated, Rao(1966a) proposed an alternative estimator to 4.2 given as

$$\hat{\tau}_{R,HT} = N\sum_{i\epsilon s}\frac{y_i}{\pi}p_i \qquad ...4.44$$

which is obtained by replacing y_i by Ny_ip_i in expression 4.2 above.

Following Rao(1966a,b), Amahia et al(1989) Grewal(1997), Rao(1993), Singh, Grewal and Joarder(2004) and Ekaette(2008), we propose the generalized alternative estimator for use in PPSWOR as:

$$\hat{\tau}_{g,HT} = \sum_{i\epsilon s}\frac{y_ip_i}{p_{i,g}^*\pi_i} \qquad …4.45$$

$p_{i,g}^*$ being the transformed size measure defined in 3.1.18 above. Again, y_i is replaced by $y_i\frac{p_i}{p_{i,g}^*}$ in 4.2 above.

This estimator reduces to $\hat{\tau}_{HT}$ when ρ=1, c>0 and $\hat{\tau}_R$ when ρ=0. When 0< ρ <1 and c>0 the following estimators are defined namely;

a) $\hat{\tau}_{ACR}$ when 0<ρ<1 and c =1;
b) $\hat{\tau}_{HT}$ when ρ=1
c) $\hat{\tau}_G$ when 0<ρ<1 and c = 1/3
d) $\hat{\tau}_{g,c}$ when c = 2, 3 and 4, which are the proposed estimators in this study.
   Again, we shall focus on those estimators defined by the c = 1, 2, 3 and 4.

4.4.2 Bias of the proposed alternative estimator in πPS sampling design

The bias of the proposed Horvitz and Thompson estimators is given as,

$$B_p(\hat{\tau}_g) = E_p(\hat{\tau}_{g,c}) - Y$$

$$= E_p\left[\sum_{i\epsilon\Omega}\frac{y_i}{\pi_i}\right] - Y$$

$$= \sum_{i\epsilon s}\sum_{i\epsilon\Omega}\frac{y_i}{\pi_i}p_i - \sum_{i\epsilon\Omega}y_i$$

$$= \sum_{i\epsilon s} \sum_{i\epsilon \Omega} \frac{y_i}{np_i} p_i - \sum_{i\epsilon\Omega} y_i$$

$$= \sum_{i\epsilon\Omega} \frac{I_{si} E(y_i) p_i}{p_{i,g}^*} - Y$$

$$= \sum_{i\epsilon\Omega} \left(\frac{I_{si} p_i}{p_{i,g}^*} - 1 \right) y_i$$

$$= \sum_{i\epsilon\Omega} \left(\frac{Np_i}{1-\rho^c+N\rho^c p_i} - 1 \right) y_i$$

$$= (1-\rho^c) N^2 COV\left(\frac{y_i}{1-\rho^c+N\rho^c p_i}, p_i \right) \qquad \ldots 4.46$$

Note that when c=0, we have the conventional estimator. When c>0, we have other alternative estimators in the linear class. With respect to correlation, the bias is an increasing function so that $\left|B(\hat{\tau}_{g,c}) - B(\hat{\tau}_{HT})\right| > 0$.

4.4.3 Variance of the proposed alternative estimator in πPS design

Theorem 4.9: the variance of the generalized alternative estimator in PPSWOR sampling is

$$V(\hat{\tau}_{g,c}) = \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1 \right) \frac{p_i^2}{p_{i,g}^{*2}} y_i^2 + \sum_{i<j}^{N} \sum \left(\frac{\pi_{ij}}{\pi_i \pi_j} - 1 \right) \frac{p_i p_j}{p_{i,g}^* p_{j,g}^*} y_i y_j \qquad \ldots 4.47$$

Proof:

$$V(\hat{\tau}_{g,c}) = E(\hat{\tau}_{g,c}^2) - E^2(\hat{\tau}_{g,c}) \qquad \ldots 4.48$$

We know that the HT estimator is unbiased as

$$E_p(\hat{\tau}_{g,c}) = E\left(\sum_{i\epsilon s} \frac{y_i}{\pi_i} \right) = \sum_{s\epsilon S} \left(\sum_{i\epsilon s} \frac{y_i}{\pi_i} \right) p(s) = \left(\sum_{i\epsilon\Omega} \frac{y_i}{\pi_i} \right) \left(\sum_{s\ni i} p(s) \right) = Y$$

As $\pi_i = \sum_{s\ni i} p(s)$ and $\pi_{ij} = \sum_{s\ni ij} p(s)$

Substituting the results above in 4.45 above, we get

$$V(\hat{\tau}_{g,c}) = \sum_{s\epsilon S} \left(\sum_{i\epsilon s} \frac{y_i^2}{\pi_i^2} \right) p(s) + \sum_{s\epsilon S} \left(\sum_{i\neq j\epsilon s} \sum \frac{y_i y_j}{p_i p_j} \right) p(s) - Y^2 \qquad \ldots 4.49$$

$$= \left(\sum_{i\epsilon s} \frac{y_i^2}{\pi_i^2}\right)\pi_i + \sum_{i\neq j\epsilon s} \sum \frac{y_i y_j}{p_i p_j}\pi_{ij} - Y^2$$

Since $Y^2 = (\sum_{i\epsilon\Omega} y_i)^2 = \sum_{i\epsilon\Omega} y_i^2 + \sum_{i\neq j=1}^{N} \sum y_i y_j$, 4.48 becomes

$$= \sum_{i\epsilon\Omega} \left(\frac{1-\pi_i}{\pi_i}\right) y_i^2 + \sum_{i\neq j\epsilon\Omega} \sum \left(\frac{\pi_{ij} - \pi_i\pi_j}{\pi_i\pi_j}\right) y_i y_j \qquad \ldots 4.50$$

4.4.4 Expected Variance of the generalized alternative πPS estimators.

Again, we consider the general super-population model defined in 4.8 above to develop the expected variance of the generalized estimator.

Theorem 4.10: The expected variance of the proposed class of alternative linear estimators is given by

$$\xi V(\hat{\tau}_{g,c}) = a \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1\right) \frac{p_i^{g+2}}{p_{i,g}^{*2}} + \beta^2 Var(\sum_{i=1}^{n} \frac{p_i^2}{\pi_i p_i^*} \qquad \ldots 4.51$$

Proof: When model based expectation is taken over (4.50) above, we have

$$\xi V(\hat{\tau}_{g,c}) = \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1\right) \frac{p_i^2}{p_{i,g}^{*2}} \xi(y_i^2) + \sum_{i<j}^{N} \sum \left(\frac{\pi_{ij}}{\pi_i\pi_j} - 1\right) \frac{p_i p_j}{p_{i,g}^* p_{j,g}^*} \xi(y_i)\xi(y_j)$$

$$= \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1\right) \frac{p_i^2}{p_{i,g}^{*2}} \left(a p_i^g + \beta^2 p_i^2\right) + \beta^2 \sum_{i<j}^{N} \sum \left(\frac{\pi_{ij}}{\pi_i\pi_j} - 1\right) \frac{p_i^2 p_j^2}{p_{i,g}^* p_{j,g}^*}$$

$$= a \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1\right) \frac{p_i^{g+2}}{p_{i,g}^{*2}} + \beta^2 Var(\sum_{i=1}^{n} \frac{p_i^2}{\pi_i p_i^*} \qquad \ldots 4.52$$

4.4.5 Comparison of expected variances of generalized alternative πPS design.

Considering $\xi V_p(\tau^*) \leq \xi\ V_p(\tau)$, let $\xi V(\hat{\tau}_{g,HT,c'})$ be the Horvitz and Thompson estimator, then;

$$\Delta_1 = \xi V(\hat{\tau}_{g,c}) - \xi V(\hat{\tau}_{g,c'})$$

$$= a\alpha_1 + \beta^2\alpha_2 \qquad \ldots 4.53$$

where

$$\alpha_1 = \sum_{i=1}^{N} \left(\frac{1}{\pi_i} - 1\right) \left[\frac{p_i^{g+2}}{p_{i,g,c}^{*2}} - \frac{p_i^{g+2}}{p_{i,g,c'}^{*2}}\right] \qquad \ldots 4.54$$

and

$$\alpha_2 = \beta^2 \left[Var\left(\sum_{i=1}^{n} \frac{p_i^2}{\pi_i p_{i,g,c}^*}\right) - Var(\sum_{i=1}^{n} \frac{p_i^2}{\pi_i p_{i,g,c'}^*})\right] \qquad \ldots 4.55$$

From 4.55, empirical evidence show that apart from the case when ρ=0 in which the two Variances are equal, if c = 1, it is clear that $\frac{p_i^{g+2}}{p_{i,g,c}^{*2}} - \frac{p_i^{g+2}}{p_{i,g,c'}^{*2}} < 0$ always. This is similar in most cases of the specification parameter *c*'s for $0 < \rho < 1$.

Similarly,

$$Var\left(\sum_{i=1}^{n} \frac{p_i^2}{\pi_i p_{i,g,c}^*}\right) - Var\left(\sum_{i=1}^{n} \frac{p_i^2}{\pi_i p_{i,g,c'}^*}\right) = Var(\sum_{i=1}^{n} b_i) - Var(\sum_{i=1}^{n} a_i)$$

where

$b_i = \frac{p_i^2}{\pi_i p_{i,g,c}^*}$ and $a_i = \frac{p_i^2}{\pi_i p_{i,g,c'}^*}$.

then,

$Var(\sum_{i=1}^{n} b_i) - Var(\sum_{i=1}^{n} a_i) < 0$ always for $0 < \rho < 1$

and

$Var(\sum_{i=1}^{n} b_i) - Var(\sum_{i=1}^{n} a_i) = 0$ when ρ=0.

It is worth to note that as c increases, $\frac{p_i^{g+2}}{p_{i,g,c}^{*2}} - \frac{p_i^{g+2}}{p_{i,g,c'}^{*2}} \ll 0$ and so

$$Var(\sum_{i=1}^{n} b_i) - Var(\sum_{i=1}^{n} a_i) \ll 0.$$

4.5 The Technique of Rao, Hartley and Cochran for n>2.

4.5.1 Introduction.

The estimator by Horvitz and Thompson(1952) is only applicable to a sample of size n=2 and requires the computational burden of the inclusion and joint inclusion probabilities respectively. To overcome this problem, especially when it is desirable to have a sample of n>2 say, then Rao-Hartley and Cochran(1962) provided a simple procedure which draws sample of size n > 2 using PPS.

Here, the population, Ω, is split into n- random groups of sizes suitably chosen N_i (i=1,2,3, …, n, $\sum N_i = N$) . in these n-groups formed, there are available positive normed-size-measure $p_i(0 < p_i < 1, \sum p_i = 1)$ which are noted and summed. From each of the n-groups formed independently, one unit is selected with PPS given the units falling in the respective groups.

Writing $$N_i = \begin{cases} \left[\frac{N}{n}\right], \ for\ i = 1,2,3,\dots,k \\ \left[\frac{N}{n}\right] + 1, for\ i = 1,2,\dots,k \end{cases} \quad \dots 4.56$$

Where k is determined by solving the equation

$$k\left[\frac{N}{n}\right] + (n-k)\left[\left[\left[\frac{N}{n}\right] + 1\right] = N\right] \quad \dots 4.57$$

So that only k-groups attain the size $\left[\frac{N}{n}\right]$ while the rest attain the size $\left[\frac{N}{n}\right] + 1$.

When selection is done under PPS, we gather these units as our sample of units selected by RHC method. Let any group be our ith group, then the value of our variable(s) chosen from the ith group is y_i. Corresponding to y_i, is the auxiliary information, x_i, upon which the normed size measure is defined.

Let Q_i be the sum of the normed-size measure of the units falling in the ith group so that $Q_i = \sum_{i=1}^{Ni} p_{ij}$, $\sum Q_i = 1$). This gives the selection probability of j in the ith group as $\frac{p_{ij}}{Q_i}$.

Define $p_{ij} = p_i$, then the estimator of population total under this strategy in PPSWR is given by

$$\hat{\tau}_{RHC} = \sum_{i=1}^{n} \frac{y_i Q_i}{p_i}, \qquad \ldots 4.58$$

with variance given by

$$V(\hat{\tau}_{RHC}) = \left[\frac{\sum_{i=1}^{n} N_i^2 - N}{N^2 - \sum_{i=1}^{n} N_i^2}\right] \sum \left[\frac{y_{ij}}{p_{ij}} - \hat{\tau}_{RHC}\right]^2 Q_i \qquad \ldots 4.59$$

4.5.2 Proposed alternative PPS estimator under RHC scheme

Now, when y and x are positively correlated, we propose a general class of estimator under linear transformation as

$$\hat{\tau}_{g,c,RHC} = \sum_{i=1}^{n} \frac{y_i Q_i}{p_i} \frac{p_i}{p_{i,g}^*} = \sum_{i=1}^{n} \frac{y_i Q_i}{p_{i,g}^*} \qquad \ldots 4.60$$

Which is realized by replacing y_i in (4.58) by $y_i p_i / p_{i,g}^*$.

4.5.3 Bias of the proposed RHC estimator.

The bias of the proposed estimator is given by

$$B\left(\hat{\tau}_{g,c,RHC}\right) = \sum_{i=1}^{n} \frac{y_i p_i}{p_{i,g}^*} - Y$$

$$= \sum_{i=1}^{n} y_i \left(\frac{p_i}{p_{i,g}^*} - 1\right) \qquad \ldots 4.61$$

Which clearly shows that the proposed generalized estimator is bias in nature especially when $\rho \neq 1$.

4.5.4 Variance of the proposed estimator.

Theorem 4.11: The variance of the proposed estimator under RHC procedure is given by $V_p\left(\hat{\tau}_{g,c,RHC}\right) = \left[\frac{\sum_{i=1}^{n} N_i^2 - N}{N^2 - \sum_{i=1}^{n} N_i^2}\right] \sum_{i=1}^{n} \left[\frac{y_i^2 Q_i}{p_{i,g}^{*2}} - \hat{\tau}_{g,RHC}^2\right]$ …4.62

Proof: By definition,

$$V_p\left(\hat{\tau}_{g,c,RHC}\right) = E_G V_c(\hat{\tau}_{g,RHC}) + V_G E_c\left(\hat{\tau}_{g,RHC}\right)$$

$$= E_G\left[\sum\sum\sum_{i\leq j<k\leq Ni}^{n} \frac{p_{ij}p_{ik}}{Q_iQ_j}\left(\frac{y_iQ_i}{p_{i,g}^*} - \frac{y_jQ_j}{p_{j,g}^*}\right)^2\right]$$

$$= E_G\left[\sum_{j\neq k}\sum p_ip_j\left(\frac{y_i}{p_{i,g}^*} - \frac{y_j}{p_{j,g}^*}\right)^2\right]$$

$$= \sum_{i\epsilon s}\frac{N_i^2-N}{N(N-1)}\left[\sum_{j\neq k}\sum p_ip_j\left(\frac{y_i}{p_{i,g}^*} - \frac{y_j}{p_{j,g}^*}\right)^2\right]$$

By Cauchy's inequality, $n\Sigma N_i^2 \geq (\Sigma N_i)^2 = N^2$. Hence $\Sigma N^2 \geq N^2/N$ and N^2 is minimal of $N_i = \left[\frac{N}{n}\right]$ for all i.

Thus, for N_i integer,

$$V\left(\hat{\tau}_{g,c,RHC}\right) = \left[\frac{\sum_{i=1}^n N_i^2 - N}{N^2 - \sum_{i=1}^n N_i^2}\right]\sum_{i=1}^n\left[\frac{y_i^2Q_i}{p_{i,g}^{*2}} - \hat{\tau}_{g,RHC}^2\right] \qquad \ldots 4.63$$

Alternatively, Rao(1966a) and Bansal and Singh(1985)'s estimator can be modified in the same manner to obtain the variance.

Now, define

$$V(\hat{\tau}_u) = \sum_{i\epsilon s}\frac{N_i^2-N}{N(N-1)}\sigma_z^2 \quad \text{where} \qquad \sigma_z^2 = \sum_{i=1}^N \frac{y_i^2}{p_i} - \left(\sum_{i=1}^N y\right)^2 \qquad \ldots 4.64$$

Under this scheme, Rao's(1966) estimator is given by

$$V(\hat{\tau}_{R,RHC}) = \frac{N-n}{(N-1)}\frac{1}{n}\left[\sum y_i^2p_i - \left(\sum y_i\right)^2\right] \qquad \ldots 4.65$$

The variance of the generalized estimator is therefore realized by replacing y_i by $y_ip_i/p_{i,g}*$ giving us:

$$V(\hat{\tau}_{g,c,RHC}) = \frac{N-n}{(N-1)}\frac{1}{n}\left[\sum\frac{y_i^2}{p_{i,g}^{*2}} - \left(\sum\frac{y_i}{p_{i,g}^*}\right)^2\right] \qquad \ldots 4.66$$

This is equivalent to

$$V(\hat{\tau}_{g,c,RHC}) = \frac{N-n}{(N-1)} Var(\hat{\tau}_{g,c}) \qquad \ldots 4.67$$

Thus, (4.66) or (4.67) can be used as our variance estimator under RHC strategy.

4.5.5 Expected Variance of RHC estimator.

Under super population model in (4.31), the expected variance is given by

$$\xi V(\hat{\tau}_{gg,c,RHC}) = \frac{N-n}{(N-1)}\frac{1}{n}\left[\sum \frac{\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum \frac{\xi(y_i)p_i}{p_{i,g}^{*}}\right)^2\right]$$

$$= \frac{N-n}{(N-1)}\frac{1}{n}\left[\sum \frac{\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum \frac{\xi(y_i)p_i}{p_{i,g}^{*}}\right)^2\right]$$

Therefore,

$$\xi V(\hat{\tau}_{g,c,RHC}) = \frac{N-n}{(N-1)} \xi Var(\hat{\tau}_{g,HH}) \qquad \ldots 4.68$$

where

$$\xi V \;\; (\hat{\tau}_{g,HH}) = \frac{1}{n}\left[\sum_{i\epsilon\Omega} \frac{I_{si}\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \left(\sum_{i\epsilon\Omega} \frac{I_i\xi(y_i)p_i}{p_i^{*}}\right)^2\right]$$

$$= \frac{1}{n}\left[\sum_{i\epsilon\Omega} \frac{\xi(y_i^2)p_i}{p_{i,g}^{*2}} - \sum_{i=1}^{N} \frac{\xi(y_i^2)p_i^2}{p_i^{*2}} - \sum\sum_{i=1}^{N} \frac{\xi(y_i y_j)p_i p_j}{p_i^* p_j^*}\right]$$

$$= \frac{a}{n}\left[\sum_{i\epsilon\Omega} \frac{p_i^{g+1}(1-p_i)}{p_{i,g}^{*2}}\right] + \frac{\beta^2}{n}\left[\sum_{i\epsilon\Omega} \frac{p_i^3}{p_{i,g}^{*2}} - \left(\sum_{i=1}^{N} \frac{p_i^2}{p_i^*}\right)^2\right] \qquad \ldots 4.69$$

4.6 Simulation of Proposed Alternative Linear Estimators in PPS Sampling Scheme Under Certain Theoretical Probability Distributions.

4.6.1 Introduction

In the previous section, a class of alternative linear estimators was proposed for use in PPS sampling schemes under a linear transformation of the selection probabilities obtained from the survey's auxiliary information. Earlier, we postulated the need to observe certain statistical properties of the study populations as they were expected to be essential requisites for specification of an estimator using the specification parameter c, which is determined by the c^{th} standardized moments of the study variable expressed as a liner function of the auxiliary variables. In this section, we utilize certain probability density functions namely; normal, chi-square, uniform and gamma distribution functions. Apart from uniform distribution, whose selection probability function is constant in the domain, the others are of the exponential family densities defined by

$$f(x,\theta) = a(\theta)b(x)\exp\{c(\theta)d(x)\} \qquad ...4.70$$

for $-\infty < x < \infty$ and for all $\theta \in \Theta$.

The main objective here is to determine if :

i. the theoretical distribution of the study populations, especially the auxiliary information have impact on the definition of our estimators;
ii. the nature of the distribution, viz-a-viz, skewed or non-skewed distribution disturb the specification of the alternative estimators and;
iii. The specification of linear alternative estimators under theoretical densities is consistent with the observed study populations.

4.6.2 The characteristics of the study distributions.

We have earlier stated that our four theoretical distributions are namely; Normal, Chi-square, Uniform and Gamma distributions.

A finite population of N = 100 is assumes in each case and the selected probabilities are simulated to realize truncated distributions which are normalized with a common denominator of N=100. Four study populations are assumed for population I, II, III and IV.

(i) x ~U(a,b) with $f(x, a, b) = \frac{1}{N}, a < x < b$; 0 elsewhere.

(ii) x ~N(μ, σ^2) with $f(x, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2, \mu > 0, \sigma > 0, x > 0$

(iii) x~χ_v^2 with $f(x, v) = \frac{1}{2^{\frac{v}{2}}\Gamma\frac{v}{2}} x^{\frac{v}{2}-1} e^{-\frac{v}{2}}, v > 0$, and

(iv) x~G(α, β) with $f(x, \alpha, \beta) = \frac{x^{\alpha-1}}{\beta^2\Gamma\alpha} \exp\{-x\beta\};\ x > 0, \alpha > 0, \beta > 0$.

The distributions represent:

i) rectangular distribution with constant probabilities;
ii) symmetric distribution with varying probabilities;
iii) skewed distributions with varying probabilities.

The table below presents the characteristics of the distribution functions being utilized.

Population	***Normal Distribution***	***Uniform Distribution***	***Chi square Distribution***	***Gamma Distribution***
I	N(6.5,3.6)	U(0,12)	χ_{14}^2	G(1,2)
II	N(15, 8.8)	U(0,30)	χ_8^2	G(1.5,5.5)
III	N(9,5.5)	U(0,17)	χ_{15}^2	G(1.5,5)
IV	N(10.5,5.92)	U(0,20)	χ_5^2	G(1,4.5)

For convenience, we shall define these densities as φ_t, t = 1, 2, 3 and 4 representing Normal, chi-square, uniform and gamma distributions respectively. Let $\varphi_{t,i}$ represent the i^{th} unit in the population, i = 1, 2, ..., N. Then the general linear transformation of the selected probabilities is defined as

$$p_{i,g}^* = \frac{1-\rho^c}{N} + \rho^c \varphi_{t,i} \ .$$

Under uniform distribution with $p_i = \varphi_{3,i}$, we have

$$p_{i,g}^{*} = \frac{1-\rho^c}{N} + \rho^c \varphi_{3,i}$$

$$= \frac{1-\rho^c}{N} + \rho^c \frac{1}{N}$$

$$= \frac{1}{N}$$

Empirically, it has been found that this is equivalent to the linear estimators when ρ=0.

Under Normal, chi-square and gamma distributions, we have

$$p_{i,g}^{*} = \frac{1-\rho^c}{N} + \rho^c \varphi_{t,i} \,, t = 1,2,4; i = 1,2, \ldots, N$$

Again, if ρ=0, we have the Rao's(1966a) estimator and if if ρ=1, we have the Hansen - Hurwitz's(1943) estimator. For 0< ρ<1 and c >0, we have the class of alternative linear estimators under investigation.

4.6.3 Simulation of selection probabilities for the study distributions

We simulate the selection probabilities $p_i = \varphi_{t,i}$ using the transformation in such a way that the simulated selection probabilities satisfy the regularity conditions of a probability normed size measure. These are shown on appendix B, C. D and E below for Normal, chi-square, uniform and gamma distributions respectively.

4.6.4 Estimation of Relative MSE of proposed alternative linear estimators under theoretical distributions .

Again, we utilize the estimator of population total, MSE, relative efficiencies using estimators defined earlier under the selection probabilities $\varphi_{t,i}$ for both sampling design and super-population model based inferences.

We define MSE($\hat{\tau}_{HH}$), MSE($\hat{\tau}_{g,1}$), MSE($\hat{\tau}_{g,2}$), MSE($\hat{\tau}_{g,3}$) and MSE($\hat{\tau}_{g,4}$) as the means square error for HHE, estimator defined by c= 1, 2, 3 and 4 respectively. Similarly, let RE($\hat{\tau}_{g,1}$), RE($\hat{\tau}_{g,2}$), RE($\hat{\tau}_{g,3}$) and RE($\hat{\tau}_{g,4}$) be the relative efficiencies using RE($\hat{\tau}_c$) as the benchmark for comparison where MSE(($\hat{\tau}_c$) and RE($\hat{\tau}_c$)

correspond with the MSE and RE of Hansen and Hurwitz Estimators and Horvitz and Thompson Estimator in PPSWR and PPSWOR sampling respectively. Then, an estimator $\hat{\tau}_{g,c}$ is relatively more efficient than another estimator $\hat{\tau}_{g,c'}$ if RE($\hat{\tau}_{g,c}$) < RE($\hat{\tau}_{g,c'}$). Specifically, an estimator with c=1 (say) is more efficient than the conventional estimator if RE($\hat{\tau}_{g,c=1}$) < RE($\hat{\tau}_{g,HH}$).

4.7 Relative Efficiency criteria.

The sensitivity of the proposed class of alternative linear estimators shall be discussed considering changing estimators defined by the specification parameter and changing correlation coefficient, ρ in both cases of positive and negative correlations. Emphasis will be drawn based on the estimate of correlation coefficient $\hat{\rho}$.

The relative percentage MSE under the sampling design will be considered. This is given by

$$RE = \left|\frac{MSE\,(\hat{\tau}_{g,c})}{MSE\,(\hat{\tau}_{HH}\,)} - 1\right| \qquad \text{...4.71}$$

Under Super-population model, the bias is very negligible as such, inference will be based on the sampling variance without taking account of the bias component. Thus, the expected relative percentage MSE and hence, expected MSE is

$$\xi RE = \left|\frac{\xi V\,(\hat{\tau}_{g,c})}{\xi V\,(\hat{\tau}_{HH}\,)} - 1\right| \qquad \text{...4.72}$$

If RE < 0 or ξRE <0, then the proposed estimator is relatively more efficient than the conventional estimator. We can also use these values to compare estimators in the same class to determine the best estimator possessing lowest percentage MSE or lowest ξMSE.

CHAPTER FIVE

DISCUSSION OF RESULTS

5.1. Introduction

In chapters three and four, we presented the various methodologies related to this study under PPS sampling design and super-population model-based inference respectively. The methodologies included the transformation of selection probabilities derived using laws of direct and inverse proportions for cases of positive and negative correlation coefficient between the study variables and selection probabilities respectively. Similarly, the generalized transformation of selection probabilities was derived. Based on these transformations and also the generalization of the selection probabilities, a class of alternative linear estimators was developed. We also derived an expression for determining approximate value of the specification parameter c under the super-population model.

In this chapter, we present the results of the developed alternative linear estimators in PPS with replacement (WR) and without replacement (WOR) sampling schemes taking cognizance of positive and negative correlation coefficient between the study and measure of size variables.

Secondly, the proposed transformations, their generalizations and also the proposed estimators are subjected to empirical studies using four populations. Population I has $\rho = 0.162$ while population II has $\rho = 0.395$. Populations III and Population IV have $\rho = -0.32$ and $\rho = -0.775$ respectively. Under the proposed inverse transformation, the resulting correlation coefficients for populations III and IV are $\rho = 0.55$ and $\rho = 0.91$ respectively. Details of the four study populations can be seeing in section 5.3 of this study.

Next, we utilized the technique of Rao, Hartley and Cochran to compare the estimators using samples of size five. Finally, we simulated the selection probabilities under normal, uniform, chi-square and gamma distributions to further investigate the behaviour of these estimators given the theoretical distributions which are either symmetrical, rectangular or skewed in nature.

It is worth to mention here that the order of correlation, otherwise, the moments coefficients of correlation coefficient is a pivot element in defining estimators for multiple characteristics as it provides a measure of relationship between selection probabilities with the study variables under linear transformation.

5.2 Results of Generalized Transformation for selection probabilities.

The result of selection probabilities and the generalized transformations of the selection probabilities derived are:

i. $p_{i,g}^{*} = \frac{1-\rho^{c}}{N} + \rho^{c} p_i$ with $p_i = \frac{x_i}{X}, X = \sum_{i \epsilon \Omega} x_i$ when $y \propto p_i$. This is appropriate when positive correlation between study and measure of size is encountered in surveys;

ii. $p_{i,g}^{*} = \frac{1-\rho^{c}}{N} + \rho^{c} p_i$ with $p_i = \frac{z_i}{Z}, Z = \sum_{i \epsilon \Omega} z_i$ and $z_i = \frac{1}{x_i}$ when $y \propto \frac{1}{x_i}$, which was proposed for use when it is clear that the study and size measures are negatively correlated.

We have c = 1,2,3 and 4 as range of the specification parameters in the sense that they are used to define appropriate estimators for the study populations. This range could also contain fractional values as in the case of Grewal(1997) and Ekaette(2008).

Specifically,

i. c = 1 and $0 < \rho < 1$ defines the Amahia-Chaubey and Rao's estimator (ACRE) defined as $\hat{\tau}_{g,c=1}$

ii. c=[0,1] and $0 < \rho < 1$ defines Ekaette' estimator (EE)

iii. c =1/3 and $0 < \rho < 1$ defines Grewals' estimator (GE) defined as $\hat{\tau}_{g,c=1/3}$

iv. for any value of c, $\rho = 1$ defines the Hansen-Hurwitz estimator (HHE) or defined as $\hat{\tau}_c$ or $\hat{\tau}_{HH}$

v. $\rho = 0$ again defines the Rao's estimator (RE) or defined as $\hat{\tau}_R$

vi. The proposed alternative estimators in the linear class are defined by c = 2, 3 and 4; they are $\hat{\tau}_{g,c=2}$, $\hat{\tau}_{g,c=3}$ and $\hat{\tau}_{g,c=4}$ respectively

vii. The generalized class of linear estimators comprises of the estimators namely; $\hat{\tau}_R$, $\hat{\tau}_{g,c=1}$ and $\hat{\tau}_{g,c=2}$, $\hat{\tau}_{g,c=3}$, $\hat{\tau}_{g,c=4}$ and $\hat{\tau}_c$.

Earlier in the methodology, it was postulated that these transformations satisfied the regularity conditions of a probability normed-size measure. Results of empirical studies conducted on four populations namely, populations I, II, III and IV having correlation coefficients of 0.162, 0.395, -0.32 and -0.77 respectively are shown on tables 1, 2, 3 and 4 below. It is worth to stress here that populations III and IV are negatively correlated and under inverse transformation, the derived correlation coefficients are 0.55 and 0.91 for populations II and IV respectively.

Table 1: ***Result of selection probabilities and Generalized selection Probabilities defining alternative Estimators in the Linear class for Population I***

X	Y	ρ	p_i	$p^*_{i,c=1}$	$p^*_{i,c=2}$	$p^*_{i,c=3}$	$p^*_{i,c=4}$
41	36	0.162	0.0806	0.0829	0.0833	0.0833	0.0833
43	47	0.162	0.0845	0.0835	0.0834	0.0833	0.0833
54	41	0.162	0.1061	0.0870	0.0839	0.0834	0.0833
39	47	0.162	0.0766	0.0822	0.0832	0.0833	0.0833
49	47	0.162	0.0963	0.0854	0.0837	0.0834	0.0833
45	45	0.162	0.0884	0.0842	0.0835	0.0834	0.0833
41	32	0.162	0.0806	0.0829	0.0833	0.0833	0.0833
33	37	0.162	0.0648	0.0803	0.0828	0.0833	0.0833
37	40	0.162	0.0727	0.0816	0.0831	0.0833	0.0833
41	41	0.162	0.0806	0.0829	0.0833	0.0833	0.0833
47	37	0.162	0.0923	0.0848	0.0836	0.0834	0.0833
39	48	0.162	0.0766	0.0822	0.0832	0.0833	0.0833
		Sum =	**1**	**1**	**1**	**1**	**1**

Table 2: ***Result of selection probabilities and Generalized selection Probabilities defining alternative Estimators in the Linear class for Population II***

X	Y	ρ	p_i	$p^*_{i,c=1}$	$p^*_{i,c=2}$	$p^*_{i,c=3}$	$p^*_{i,c=4}$
3	11	0.39	0.0133	0.0255	0.0303	0.0321	0.0329
4	7	0.39	0.0178	0.0273	0.0310	0.0324	0.0330
5	9	0.39	0.0222	0.0290	0.0316	0.0327	0.0331
8	8	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
12	8	0.39	0.0533	0.0411	0.0364	0.0345	0.0338
11	9	0.39	0.0489	0.0394	0.0357	0.0343	0.0337
8	8	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
9	12	0.39	0.0400	0.0359	0.0343	0.0337	0.0335
11	10	0.39	0.0489	0.0394	0.0357	0.0343	0.0337
10	9	0.39	0.0444	0.0377	0.0350	0.0340	0.0336
8	8	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
9	14	0.39	0.0400	0.0359	0.0343	0.0337	0.0335
7	12	0.39	0.0311	0.0325	0.0330	0.0332	0.0333
8	10	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
8	10	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
5	10	0.39	0.0222	0.0290	0.0316	0.0327	0.0331
6	9	0.39	0.0267	0.0307	0.0323	0.0329	0.0332
3	5	0.39	0.0133	0.0255	0.0303	0.0321	0.0329
3	7	0.39	0.0133	0.0255	0.0303	0.0321	0.0329
9	9	0.39	0.0400	0.0359	0.0343	0.0337	0.0335
6	6	0.39	0.0267	0.0307	0.0323	0.0329	0.0332
7	12	0.39	0.0311	0.0325	0.0330	0.0332	0.0333
8	9	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
8	6	0.39	0.0356	0.0342	0.0337	0.0335	0.0334
9	9	0.39	0.0400	0.0359	0.0343	0.0337	0.0335
11	11	0.39	0.0489	0.0394	0.0357	0.0343	0.0337
11	10	0.39	0.0489	0.0394	0.0357	0.0343	0.0337
10	14	0.39	0.0444	0.0377	0.0350	0.0340	0.0336
5	8	0.39	0.0222	0.0290	0.0316	0.0327	0.0331
3	7	0.39	0.0133	0.0255	0.0303	0.0321	0.0329
		Sum =	1	1	1	1	1

We observe here that the population correlation coefficient is 0.395 while the selection probabilities p_i and the generalized selection probabilities; $p^*_{i,c=1}$, $p^*_{i,c=2}$, $p^*_{i,c=3}$ and $p^*_{i,c=4}$ do satisfy the required conditions of a probability size measure.

Table 3: Result of selection probabilities and Generalized selection Probabilities defining alternative Estimators in the Linear class for Population III

X	y	ρ	p_i	$p^*_{i,c=1}$	$p^*_{i,c=2}$	$p^*_{i,c=3}$	$p^*_{i,c=4}$
100	3	0.55	0.0279	0.0418	0.0495	0.0537	0.0560
88	8	0.55	0.0317	0.0439	0.0506	0.0543	0.0563
20	9	0.55	0.1396	0.1033	0.0833	0.0723	0.0662
17	11	0.55	0.1643	0.1168	0.0907	0.0764	0.0685
60	5	0.55	0.0465	0.0521	0.0551	0.0568	0.0577
77	9	0.55	0.0363	0.0464	0.0520	0.0551	0.0568
51	5	0.55	0.0548	0.0566	0.0576	0.0581	0.0585
69	4	0.55	0.0405	0.0487	0.0533	0.0558	0.0571
66	6	0.55	0.0423	0.0497	0.0538	0.0561	0.0573
77	9	0.55	0.0363	0.0464	0.0520	0.0551	0.0568
68	2	0.55	0.0411	0.0491	0.0535	0.0559	0.0572
36	4	0.55	0.0776	0.0691	0.0645	0.0619	0.0605
74	4	0.55	0.0377	0.0472	0.0524	0.0553	0.0569
33	5	0.55	0.0846	0.0730	0.0666	0.0631	0.0612
54	6	0.55	0.0517	0.0549	0.0567	0.0576	0.0582
55	6	0.55	0.0508	0.0544	0.0564	0.0575	0.0581
77	6	0.55	0.0363	0.0464	0.0520	0.0551	0.0568
		Sum =	**1**	**1**	**1**	**1**	**1**

For the target population above, the population correlation coefficient is -0.322. However, under the inverse transformation, the resulting value of correlation coefficient is 0.55. The selection probabilities and the generalized selection probabilities; $p^*_{i,c=1}$, $p^*_{i,c=2}$, $p^*_{i,c=3}$ and $p^*_{i,c=4}$ also do satisfy the required conditions of a probability size measure.

Table 4: Result of selection probabilities and Generalized selection Probabilities defining alternative Estimators in the Linear class for Population IV

X	y	ρ	p_i	$p^*_{i,c=1}$	$p^*_{i,c=2}$	$p^*_{i,c=3}$	$p^*_{i,c=4}$
6.8	20	0.91	0.00324	0.00745	0.01128	0.01476	0.01793
6.2	23	0.91	0.00355	0.00773	0.01154	0.015	0.01815
5.5	38	0.91	0.004	0.00814	0.01191	0.01534	0.01846
0.85	86	0.91	0.0259	0.02807	0.03004	0.03184	0.03347
0.71	92	0.91	0.03101	0.03272	0.03427	0.03569	0.03698
9	16	0.91	0.00245	0.00673	0.01062	0.01416	0.01739
1.4	81	0.91	0.01573	0.01881	0.02162	0.02417	0.0265
4.5	53	0.91	0.00489	0.00895	0.01265	0.01601	0.01907
3.8	42	0.91	0.00579	0.00977	0.01339	0.01669	0.01969
2.1	62	0.91	0.01048	0.01404	0.01728	0.02022	0.0229
4.85	39	0.91	0.00454	0.00863	0.01235	0.01574	0.01883
3.197	35	0.91	0.00689	0.01077	0.0143	0.01751	0.02043
0.443	87	0.91	0.04968	0.0497	0.04973	0.04976	0.04978
0.468	91	0.91	0.04704	0.0473	0.04755	0.04777	0.04797
0.59	84	0.91	0.03732	0.03846	0.0395	0.04044	0.0413
0.339	75	0.91	0.06495	0.0636	0.06238	0.06127	0.06025
0.161	54	0.91	0.13642	0.12865	0.12157	0.11513	0.10927
0.787	64	0.91	0.02797	0.02995	0.03175	0.0334	0.03489
0.069	26	0.91	0.31801	0.29389	0.27194	0.25197	0.23379
0.11	100	0.91	0.20015	0.18663	0.17434	0.16315	0.15296
		Sum =	**1**	**1**	**1**	**1**	**1**

From tables 1, 2, 3 and 4 above, it is clear that the linear transformations p_i and hence, the generalized transformation p_i^* all satisfied the regularity conditions of a probability normed-size measure, namely:

i. $0 < p_i < 1$;

ii. $\sum_{i\epsilon\Omega} p_i = 1$;

iii. $0 < p^*_{i,g} < 1$ and

iv. $\sum_{i\epsilon\Omega} p^*_{i,g} = 1$.

These results are consistent for all the cth moment utilized in this study and hence the specification range c = [1, 4]. The implication of this range is that it can be utilized in defining a class of linear estimators defined by the moments providing an optimum estimator for a target population.

5.3 Statistical Properties of the study Populations

In this section, the statistical properties of the study populations are presented. We have earlier postulated that these characteristics are determined by the expectation of the linear function as well as the expectation of the c^{th} standardized moment of the study variate, y under the linear framework. The results in tables 5, 6, 7 and 8 below show the peculiar characteristics of the target populations and these include, the coefficients of variation, determination, skewness and kurtosis of the study variate and the measure of size variate as they relate with moment in correlation coefficient.

Table 5: Statistical Properties of Population I

Parameter	X	Y	ε	Ratio	$R^2\sigma_{\varepsilon,y}$	$R^3\sigma_{\varepsilon,y}$	$R^4\sigma_{\varepsilon,y}$
Mean	42.4167	41.5000	0.0000				
Median	41.0000	41.0000	-0.4821				
Maximum	54.0000	48.0000	7.0180				
Minimum	33.0000	32.0000	-9.2852				
Std. Dev.	5.6642	5.3001	5.2301	**0.0262**	0.9738	0.9609	0.9482
coefficient of skewness	0.4551	-0.2506	-0.1897	**-0.5507**			
coefficient of kurtosis	2.8302	1.8306	1.8763	**0.6468**			
correlation coefficient	**0.1620**						
Coefficient of determination	**0.0262**						
coefficient of variation	0.1335	0.1277		**0.5227**			
Observations	12.0000	12.0000	12.0000				

5.3.1 Description of statistical Properties of Population I

It is observed from table 5 above that $CV_x = 0.1335 > CV_y = 0.1277$. By the propositions, the required condition for specifying c= 1 is when $\rho^1 = \frac{CV_x}{CV_y} < 1$. Under linear transformation, $\rho^1 = 0.162$. Furthermore, $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1$ and also $\rho^4 = \frac{K_y}{K_x} < 1$ are expected. By (3.1.27) above, it is clear that since these conditions are fulfilled with $\rho \to 0$ and hence, $\rho^2 \to 0$. Higher moment may not be required in defining the estimator for this population. Therefore, c = 1 could be sufficient in defining estimator for this population.

Table 6: Statistical Properties of Population II

Parameter	X	Y	ε	Ratio	$R^2\sigma_{\varepsilon,y}$	$R^3\sigma_{\varepsilon,y}$	$R^4\sigma_{\varepsilon,y}$
Mean	7.500	9.233	0.000				
Median	8.000	9.000	-0.411				
Maximum	12.000	14.000	4.283				
Minimum	3.000	5.000	-3.394				
Std. Dev.	2.688	2.192	2.014	**0.156**	0.844	0.775	0.712
Skewness	-0.276	0.356	0.537	**-1.290**			
Kurtosis	2.069	2.866	2.583	**1.385**			
correlation coefficient	**0.395**						
Coefficient of determination	**0.156**						
coefficient of variation	**0.358**	0.237	6047756.757	**0.7553**			
Observations	30.000	30.000	30.000				

5.3.2 Description of statistical Properties of Population II

Here, in table 6, $\hat{\rho} = 0.395 < 1$, $\rho^2 = 0.156 < 1$.However, $|\rho^3| > 1$ thus violating the postulated conditions required by ρ^c. This suggests that some other estimators could perform better than that defined by c = 1. By this result, the correlation coefficient is not too weak but somewhat “moderate”. However, under linear framework, $\rho^1 = \hat{\rho} < 1$, $\rho^2 < 1$, $\rho^3 > 1$ and $\rho^4 > 1$. At this point, c > 1 could be most suitable for defining the estimator.

Table 7: Statistical Properties of Population III

Parameter	X	Y	ε	Ratio	$R^2\sigma_{\varepsilon,y}$	$R^3\sigma_{\varepsilon,y}$	$R^4\sigma_{\varepsilon,y}$
Mean	60.11765	6	-0.00000118				
Median	66	6	-0.20886				
Maximum	100	11	3.57636				
Minimum	17	2	-3.7309				
Std. Dev.	23.08106	2.44949	2.319285	**0.103**	**0.897**	**0.849**	**0.804**
Skewness	-0.39924	0.420813	0.350907	**-1.054**			
Kurtosis	2.424363	2.346354	1.976402	**0.968**			
correlation coefficient	-0.320						
Coefficient of determination	0.103						
coefficient of variation	0.384	0.408	-1965495.73	**0.47059**			
Observations	17.000	17.000	17.000				

5.3.3 Description of statistical Properties of Population III

For population III, the results on table 7 above shows that, $\hat{\rho} = -0.32$, $\rho^2 = 0.103$ $|\rho^3| > 1$ and $|\rho^4| < 1$. The correlation coefficient is negative and moderate, and the required assumption is violated. Therefore, the range 2≤ c≤4 provides the appropriate specification defining the estimator for this population, even as the transformed $\hat{\rho}$=0.55, which is not weak or high, but about average or what we may call "moderate" value of ρ.

Table 8: Statistical Properties of Population IV

Parameter	X	Y	E	Ratio	$R^2\sigma_{\varepsilon,y}$	$R^3\sigma_{\varepsilon,y}$	$R^4\sigma_{\varepsilon,y}$
Mean	58.400	2.594	0.000				
Median	58.000	1.125	0.370				
Maximum	100.000	9.000	3.160				
Minimum	16.000	0.069	-5.005				
Std. Dev.	27.358	2.704	1.711	**0.600**	0.400	0.253	0.160
Skewness	**-0.059**	**0.886**	-1.191	**-0.067**			
Kurtosis	**1.600**	**2.601**	5.208	**0.615**			
correlation coefficient	**-0.775**						
Coefficient of determination	**0.601**						
coefficient of variation	**0.468**	**1.043**		**0.2244**			
Observations	20.000	20.000	20.000				

5.3.4 Description of statistical Properties of Population IV

For population IV, it is evidenced on table 8 that, $\hat{\rho} = -0.775$, $\rho^2 < 1$, $\rho^3 < 1$ and $\rho^4 < 1$ satisfying all conditions. However, the correlation coefficient is negative and high. Under linear transformation, $\hat{\rho} = 0.91$ which is a strong positive correlation coefficient. We have earlier speculated in 3.1.36 that $(1-\rho^2)^2 K_\varepsilon \longrightarrow 0$ when ρ=1 hence, $\rho^4 K_x$<1. Thus, as ρ→1, c →4 would provide appropriate estimator of the parameters of interest. In other word, we expect that the appropriate estimator be found between c=3 and c=4.

5.4 Estimate of bias of the alternative linear Estimators.

It has been shown in (3.2.4) that the alternative estimators are biased. It is therefore, necessary to discuss the magnitude and sign of the bias which are shown on tables 9, 10, 11 and 12 for populations 1, 2, 3 and 4 respectively for PPSWR sampling design.

Table 9: Estimate of design-based Bias, $B(\hat{\tau}_{g,c})$ of alternative estimators as compared with that of HHE for population I under PPSWR sampling design.

Rho	$B(\hat{\tau}_{HHE})$	$B(\hat{\tau}_{g,1})$	$B(\hat{\tau}_{g,2})$	$B(\hat{\tau}_{g,3})$	$B(\hat{\tau}_{g,4})$
0.000	0.000	1.261	1.261	1.261	1.261
0.100	0.000	0.419	1.169	1.252	1.260
0.162	0.000	**-0.020**	1.024	1.222	1.255
0.500	0.000	-1.328	-0.536	0.234	0.715
0.900	0.000	-0.577	-0.962	-1.201	-1.330
1.000	0.000	0.000	0.000	0.000	0.000

Table 10: Estimate of design-based Bias, $B(\hat{\tau}_{g,c})$ of alternative estimators as compared with that of HHE for population II under PPSWR sampling design.

Rho	$B(\hat{\tau}_{HHE})$	$B(\hat{\tau}_{g,1})$	$B(\hat{\tau}_{g,2})$	$B(\hat{\tau}_{g,3})$	$B(\hat{\tau}_{g,4})$
0.000	0.000	9.000	9.000	9.000	9.000
0.100	0.000	5.183	8.591	8.959	8.996
0.395	0.000	-2.696	3.303	6.577	8.015
0.500	0.000	-4.279	0.554	4.322	6.544
0.900	0.000	-2.965	-4.513	-5.193	-5.319
1.000	0.000	0.000	0.000	0.000	0.000

Table 11: Estimate of design-based Bias, $B(\hat{\tau}_{g,c})$ of alternative estimators as compared with that of HHE for population III under PPSWR sampling design.

Rho	$B(\hat{\tau}_{HHE})$	$B(\hat{\tau}_{g,1})$	$B(\hat{\tau}_{g,2})$	$B(\hat{\tau}_{g,3})$	$B(\hat{\tau}_{g,4})$
1.000	0.000	0.000	0.000	0.000	0.000
0.9	0.0	-12.4	-20.3	-24.8	-26.9
0.5	0.0	-25.2	**-3.2**	19.0	33.9
0.3	0.0	-12.1	24.1	42.1	48.8
0.1	0.0	24.6	48.9	51.7	52.0
0.0	0.0	52.0	52.0	52.0	52.0

Table 12: Estimate of design-based Bias, $B(\hat{\tau}_{g,c})$ of alternative estimators as compared with that of HHE for population IV under PPSWR sampling design.

Rho	$B(\hat{\tau}_{HHE})$	$B(\hat{\tau}_{g,1})$	$B(\hat{\tau}_{g,2})$	$B(\hat{\tau}_{g,3})$	$B(\hat{\tau}_{g,4})$
1.000	0.000	0.000	0.000	0.000	0.000
0.9	0.0	-159.0	-218.4	-248.4	**-264.4**
0.8	0.0	-241.3	-273.5	-267.7	-245.8
0.5	0.0	-271.5	-201.4	-109.5	-34.2
0.1	0.0	-82.7	56.6	76.0	78.0
0.0	0.0	78.2	78.2	78.2	78.2

5.4.1 Description of the Bias of the estimators in PPSWR sampling schemes.

The alternative estimators described by the cth moment, c = 1,2,3 and 4 in correlation coefficient presented the following bias for the alternative estimators in the linear class as shown in tables 9, 10, 11 and 12 above.

The bias of the proposed estimators $B(\hat{\tau}_{g,c})$, c = 1,2,3,4 is same when ρ=0 and is more than the bias of the conventional estimator, $B(\hat{\tau}_c) = 0$ for all the study populations as shown in tables 9 to 12 above. It is worth to note here that the said bias actually occurs when $0 \leq \rho < 1$. When ρ=1, the bias of all the estimators including the Hansen-Hurwitz estimator is zero. That is, $B(\hat{\tau}_{g,c}) = B(\hat{\tau}_c)$. Thus all estimators converge at this point.

It can be seen from table 9 above that in population 1, minimum bias is attained at ρ=0.162 by the estimator defined by c = 1. Thus, $|B(\hat{\tau}_{g,c=1})| < |B(\hat{\tau}_{g,c})|, c = 2,3,4$. Again, for population 2, minimum bias is attained at ρ=0.395 by the estimator defined by c=1, that is, $|B(\hat{\tau}_{g,c=1})| < |B(\hat{\tau}_{g,c})|, c = 2,3,4$ as contained in table 10. For populations 3 and 4 which have negative correlation coefficient, transformation is required and so, minimum bias can only be attained at $\rho = \hat{\rho}$ or its neighbourhood as shown in tables 11 and 12 respectively. Thus, for population 3, bias is minimized by the estimator defined by c = 2, that is, $|B(\hat{\tau}_{g,c=2})| < |B(\hat{\tau}_{g,c})|, c = 1,3,4$ when ρ= |0.32| and $|B(\hat{\tau}_{g,c=2})| < |B(\hat{\tau}_{g,c})|, c = 1,3,4$ when ρ= |0.55| while in the case of population 4 bias is minimized at c = 4, that is $|B(\hat{\tau}_{g,c=4})| < |B(\hat{\tau}_{g,c})|, c = 1,2,3$ at ρ = |0.50| and c=1/4 when ρ= |0.91|.

Table 13: Approximate values of c at g = 0,1,2 for populations I, II, III and IV

Population	Rho	N	g=0		g=1		g=2		Estimate of c
			Min p_i	Max p_i	Min p_i	Max p_i	Min p_i	Max p_i	
I	0.162	12	0	2	0	2	0	1	**1**
II	0.395	30	0	2	0	4	0	2	**2**
III	-0.32 *(0.55)*	17	0	3	0	3	0	2	**2**
IV	-0.775 *(0.91)*	20	0	13	0	6	0	8	**4**

- Values in italics are the transformed correlation coefficient.

Table 13 presents the estimated values of c for the four study populations using the derived expression in (4.27) above. It can be clearly seen that for population I, the values of c falls between 0 and 2 at g =0 and 1. However, at g=2, the estimated value of c is 1, that is, c=1.

For population II, the values of c falls between 0 and 2 at g =0, 0 and 4 at g=1 and 0 and 2 at g=2 while for population III, the values of c falls between 0 and 3 at g =0, 0 and 3 at g=1 and 0 and 2 at g=2 . For population IV, the values of c falls between 0 and 13 at g =0, 0 and 6 at g=1 and 0 and 8 at g=2. It is therefore worthy to note that the ceiling of the value of c is expected to be 4 as higher moments are assumed to be covered in the ceiling value with c = 4.

5.4.2 Design-based Relative MSE of the proposed estimators compared with Hansen-Hurwitz estimator

We utilized the conventional Hansen and Hurwitz Estimator (HHE) as the denominator in order to compare the performance of the proposed alternative linear estimators using the relative efficiency criteria defined by

$RE\left(\hat{\tau}_{g,c} \backslash \hat{\tau}_{HHE}\right) = 0$ or $MSE(\hat{\tau}_{g,c}) = MSE(\hat{\tau}_{HH})$

Any alternative estimator is relatively more efficient than HHE in terms of minimum variance and hence, MSE if and only if

$RE\left(\hat{\tau}_{g,c} \backslash \hat{\tau}_{HHE}\right) < 1\ (100\%)$

or

$RE\left(\hat{\tau}_{g,c} \backslash \hat{\tau}_{HHE}\right) < RE(\hat{\tau}_{HH} \backslash \hat{\tau}_{HHE})$

otherwise, HHE is relatively the most efficient estimator for the study population.

The efficiency of the proposed estimator given the conventional HHE for the PPS sampling design is presented in Tables 14, 15, 16 and 17 above for populations I to IV respectively.

Table 14: Design-based Relative efficiency of alternative estimators as compared with HHE for population I (measured by $\mathrm{RE}(\hat{\tau}_{g,c} \backslash \hat{\tau}_{HH}) = \frac{MSE(\hat{\tau}_{g,c})}{MSE(\hat{\tau}_{HH})} < \mathbf{1}$ ***)***

Rho	$\mathrm{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_1 \backslash \hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_2 \backslash \hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_3 \backslash \hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_4 \backslash \hat{\tau}_{HH})$
0.000	100.0	57.0	57.0	57.0	57.0
0.100	100.0	56.1	56.9	57.0	57.0
0.162	100.0	**56.1**	56.7	57.0	57.0
0.500	100.0	64.2	56.9	56.0	56.3
0.900	100.0	90.4	82.9	76.9	72.2
1.000	100.0	100.0	100.0	100.0	100.0

From table 14 above, it is clear that $\mathrm{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=1}) < \mathrm{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c})$, c = 2,3,4 and also, $\mathrm{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=1}) < \mathrm{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{HH})$ for population I at $\rho = 0.162$ in terms of minimum variance. Thus, the estimator defined by c = 1 has minimum percentage relative MSE of 56.1% and this further confirm our postulation that the specification, c = 1 is only possible when $\rho \to 0$. $RE(\hat{\tau}_c)$

Table 15: Design-based Relative efficiency of alternative estimators as compared with HHE for population II (measured by $\mathrm{RE}(\hat{\tau}_{g,c}\backslash\hat{\tau}_{HH}) = \frac{\boldsymbol{MSE(\hat{\tau}_{g,c})}}{\boldsymbol{MSE(\hat{\tau}_{HH})}} < \mathbf{1}$ ***)***

Rho	$\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_1\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_2\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_3\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_4\backslash\hat{\tau}_{HH})$
0.000	100.0	35.2	35.2	35.2	35.2
0.100	100.0	31.6	34.7	35.1	35.2
0.395	100.0	30.8	**30.4**	32.8	34.1
0.500	100.0	33.8	29.5	31.0	32.7
0.900	100.0	73.7	58.4	48.8	42.4
1.000	100.0	100.0	100.0	100.0	100.0

For population II, $\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=2})$ < $\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c})$, c = 1,3,4 and also, $\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=2})$ < $\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{HH})$ so that the estimator defined by c = 2 with minimum percentage relative MSE of 30.4% performed better than all the competing estimators including the conventional estimator at $\rho = 0.395$ as shown on table 15. Again, we have postulated that this is possible when ρ<1, $\rho^2 < 1$ and $Cv_x < Cv_y$. It is also clear that as ρ shift upwards, say, ρ → 0.5 rather than ρ→0, c=2 is best specified for a target population.

Table 16: Design-based Relative efficiency of alternative estimators as compared with HHE for population III (measured by $\text{RE}(\hat{\tau}_{g,c} \backslash \hat{\tau}_{HH}) = \frac{MSE(\hat{\tau}_{g,c})}{MSE(\hat{\tau}_{HH})} < 1$ *)*

Rho	$\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{HH})$	$\text{RE}(\hat{\tau}_1 \backslash \hat{\tau}_{HH})$	$\text{RE}(\hat{\tau}_2 \backslash \hat{\tau}_{HH})$	$\text{RE}(\hat{\tau}_3 \backslash \hat{\tau}_{HH})$	$\text{RE}(\hat{\tau}_4 \backslash \hat{\tau}_{HH})$
0	100	44.2	44.2	44.2	44.2
0.1	100	39.3	43.4	44.1	44.1
0.3	100	40.4	**39.2**	42	43.4
0.5	100	48	**38.8**	**38.8**	40.5
0.9	100	84.3	73	64.7	58.4
1	100	100	100	100	100

Population III is analysed under two conditions of correlation, that is, at the actual correlation coefficient of $\hat{\rho} = |-0.32|$ and the correlation coefficient of $\rho = 0.55$ realized after transformation of the measure of size variable. Examining the results on table 16 above, it is observed that the estimator defined by c = 2 with minimum percentage relative MSE of 39.24% performed better than all the competing estimators including the conventional estimator at $\hat{\rho} = |-0.32|$. The result is the same when $\rho = 0.55$ is considered. However, at $\rho = 0.55$, minimum percentage relative MSE is obtained at two points namely, c=2 and c=3 as shown on table 16. Thus, $\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=2})$= $\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=3})$ < $\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=1})$ and $\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=3})$ < $\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{g,c=4})$ < $\text{RE}(\hat{\tau}_{HH} \backslash \hat{\tau}_{HH})$.

Based on this result, we can infer as follows:

i. When 0.3 < ρ < 0.5 or neighbourhood, the estimator defined by c =2, that is, $\hat{\tau}_{g,c=2}$ would be relatively more efficient than all other estimators;

ii. When ρ is slightly greater than 0.5, the estimator changes from c=2 to c=3 as evidenced in our study which c = 3 at ρ = 0.55.

This suggests that ρ = 0.55 is perhaps a boundary point for which two estimators defined by c=2 and c=3 performed best for population III.

Table 17: Design-based Relative efficiency of alternative estimators as compared with HHE for population IV (measured by $\mathrm{RE}(\hat{\tau}_{g,c}\backslash\hat{\tau}_{HH}) = \frac{MSE(\hat{\tau}_{g,c})}{MSE(\hat{\tau}_{HH})} < 1$ ***)***

Rho	$\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_1\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_2\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_3\backslash\hat{\tau}_{HH})$	$\mathrm{RE}(\hat{\tau}_4\backslash\hat{\tau}_{HH})$
0	100	12.5	12.5	12.5	12.5
0.1	100	12.2	12.2	12.5	12.5
0.5	100	22.6	15.8	12.6	11.7
0.8	100	32.9	25.5	21.6	**18.8**
0.9	100	48.5	36.9	31.6	**28.5**
1	100	100	100	100	100

Again, for population IV, the percentage relative is analysed at the correlation coefficient of $\hat{\rho} = |-0.775|$ and the correlation coefficient of $\rho = 0.91$ realized after transformation of the measure of size variable. Looking at results on table 17 above, it is clear that $\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=4}) < \mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c})$, c = 1,2,3 and also, $\mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=4}) < \mathrm{RE}(\hat{\tau}_{HH}\backslash\hat{\tau}_{HH})$ for population IV so that the estimator defined by c = 4 is relatively more efficient with percentage relative MSE of 18.8% when $|\rho = -0.775|$ and 28.5% when $\rho = 0.91$. It is also clear from results on table 16 that estimators defined by c =4 is relatively more efficient than all other estimators for population IV.

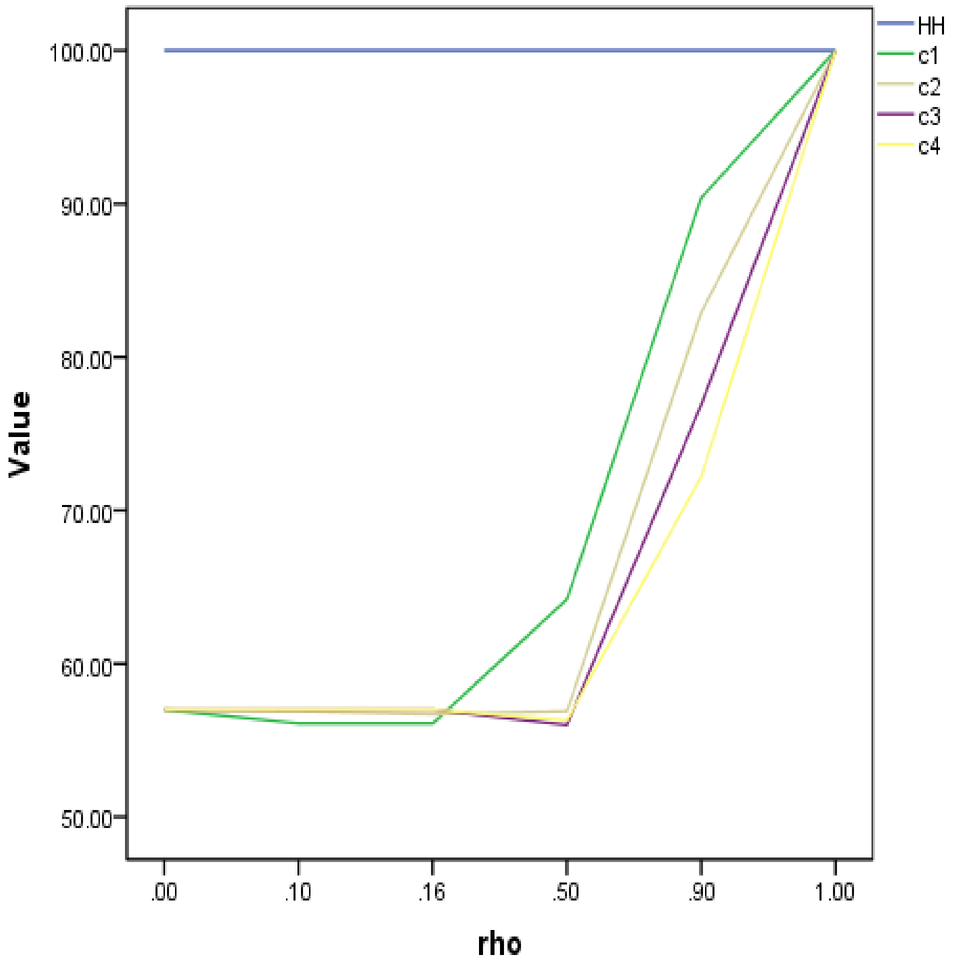
By these results, it will be convenient to state that the specification parameter of an estimator c, changes with ρ. Thus,

i. As ρ→0, estimator defined by c =1 would be appropriate;
ii. As ρ→0.5, estimator defined by c = 2 would be preferred;
iii. As 0.5<ρ<.75, estimator defined by c=3 would be preferred while
iv. As ρ→1, estimator defined by c = 4 would be preferred.

The results described above are further displayed on figures 1 to 4 below, showing the relative performances of the proposed estimators in the parameter space, $\hat{\tau}_{g,c}$ c=1,2,3 and 4.

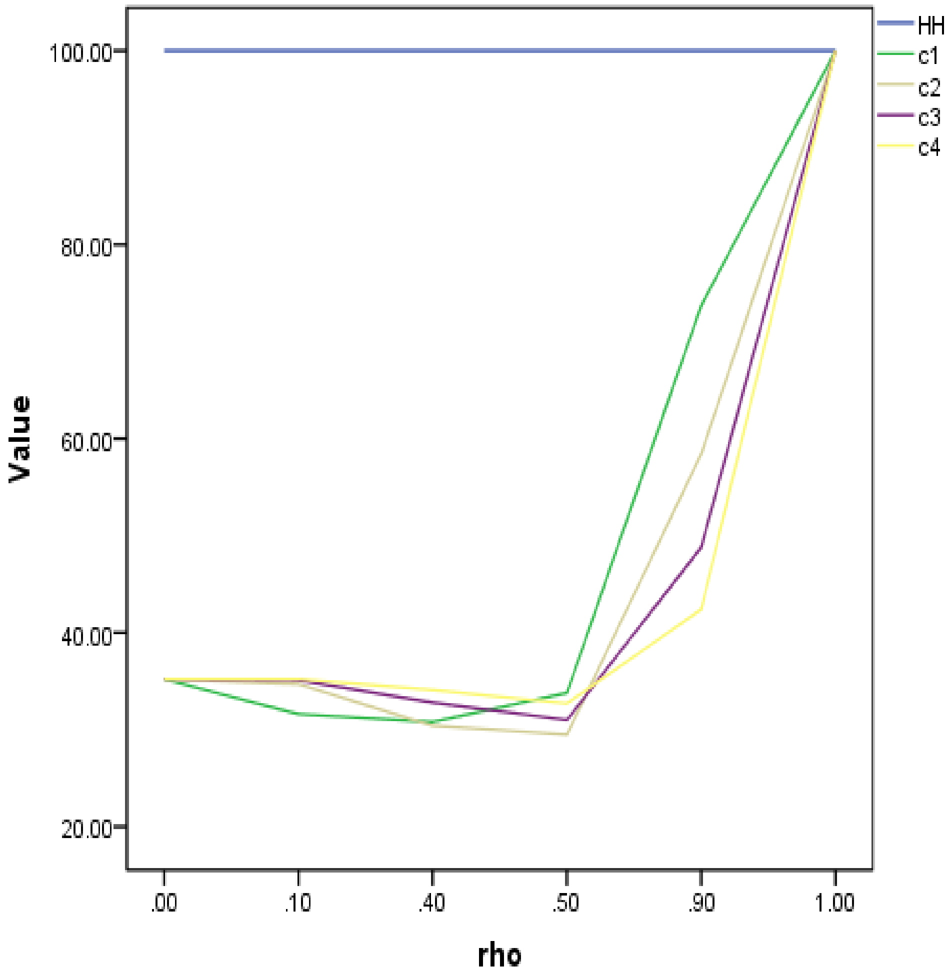
Figures 1 to 4 below presents the graphical view of the alternative estimators as compared with the Hansen and Hurwitz estimator for the four study populations.

Fig. 1: Graph of Relative MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population I



On figure 1 above, the behaviour of the estimators in the parameters space with respect to MSE defined by the relative efficiency (RE) is presented. Again, It is clear that the estimator defined by c = 1 (with green coloured line), otherwise, the ACRE is uniformly most efficient (UME) estimator for population I when $0<\rho\leq0.162$ and neighbourhood. However, when $0.16<\rho\leq0.50$ and its neighbourhood, estimators defined by c = 2 and c = 3 performed equally better than other estimators. However, for $\rho > 0.50$, the estimator defined by c = 4 performed better than all other estimators.

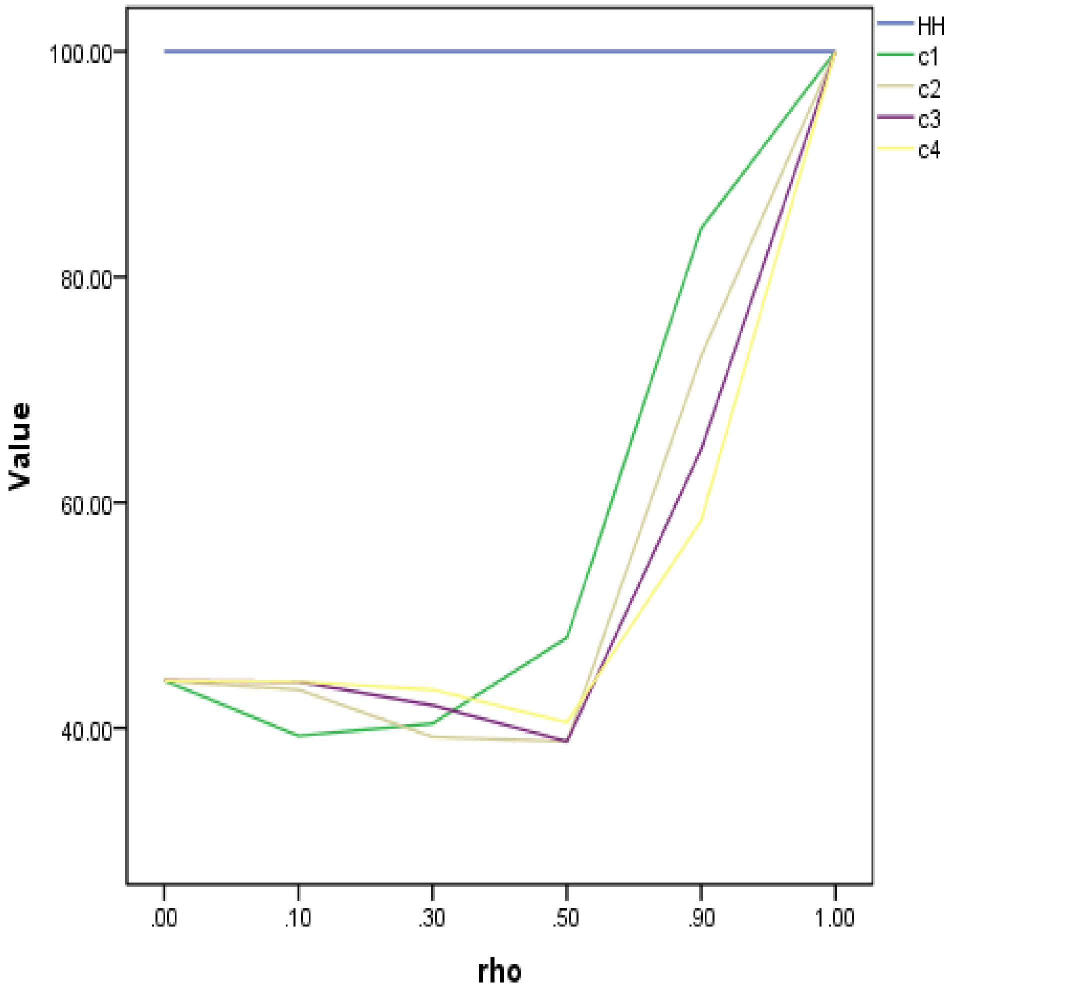
Fig. 2: Graph of Relative MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population II



For population II, results displayed on figure 2 above that for the values of ρ =0.395 and its neighbourhood, the estimator defined by c=2 (in grey colour) is most efficient for the $0.39 <\rho<0.5$.

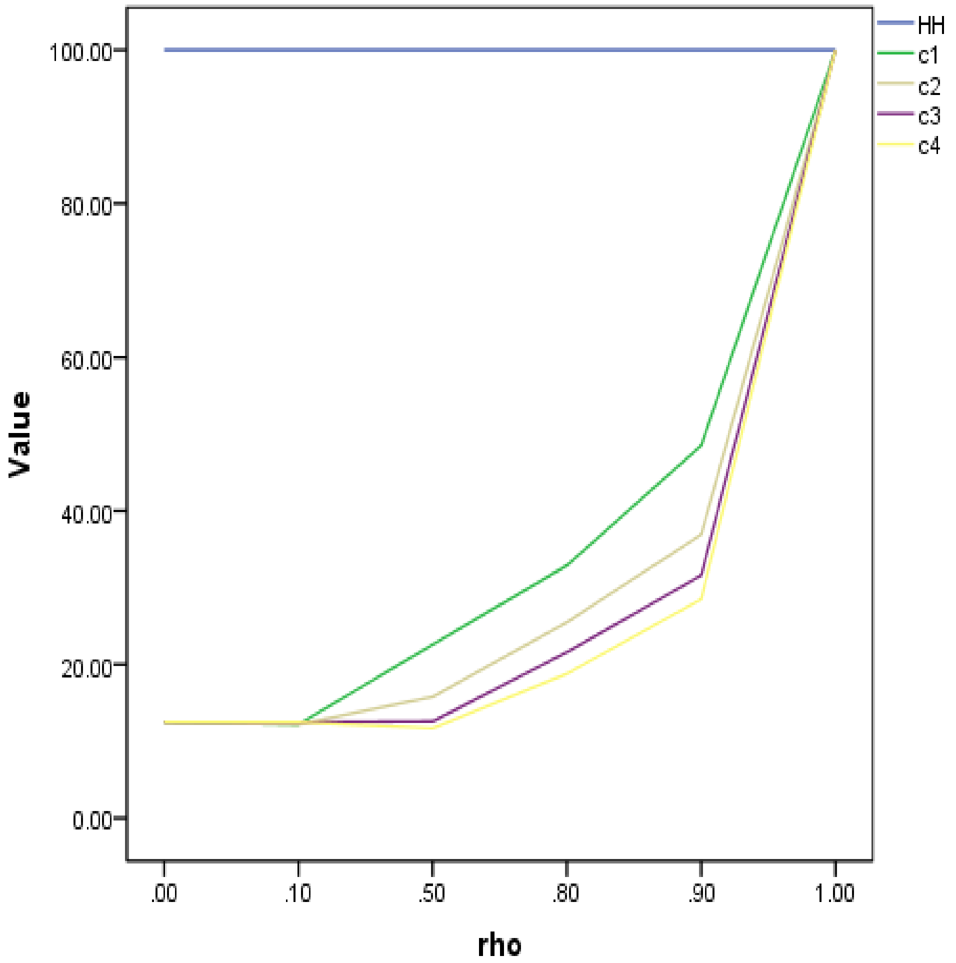
It is also noticeable here that the estimator defined by c = 1 performed better than all other estimators if 0.01 <ρ<0.39 were assumed for this study population. As ρ>0.5, the estimator defined by c = 4 performed better than all other estimators.

Fig. 3: Graph of Relative MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population III



For population III, the result displayed on figure 3 shows that the estimator defined by c = 1 is most efficient for 0 <ρ<0.28 or neighbourhood. However, when 0.30 <ρ<0.50 or its neighbourhood, estimator defined by c = 2 (with grey lines) performed best than all other estimators. As ρ>0.60, the estimator defined by c = 4 performed better than all other estimators in this class. Similarly, under linear transformation, two estimators namely, c=2 and c=3 performed equally well for population III under the derived value of ρ=0.55

Fig. 4: Graph of Relative MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population IV



In the case of population IV, the results displayed on figure 4 above shows that the estimator defined by c = 4 (with yellow line) is relatively most efficient than all other estimators including the conventional estimator throughout the parameter space defined by the correlation coefficient ρ. It is worthy to state here that all the values of correlation considered for this population is very high.

A closer look at figures 1 to 4 shows that the efficiency of estimators is changing along moments in the correlation coefficient. Thus, for populations that are weakly correlated, the estimators defined by c = 1 is sufficient. It is also noticeable that there are certain points in the moments in ρ in which two estimators could perform best and these points are the adjoining points, otherwise, boundary point between two estimators.

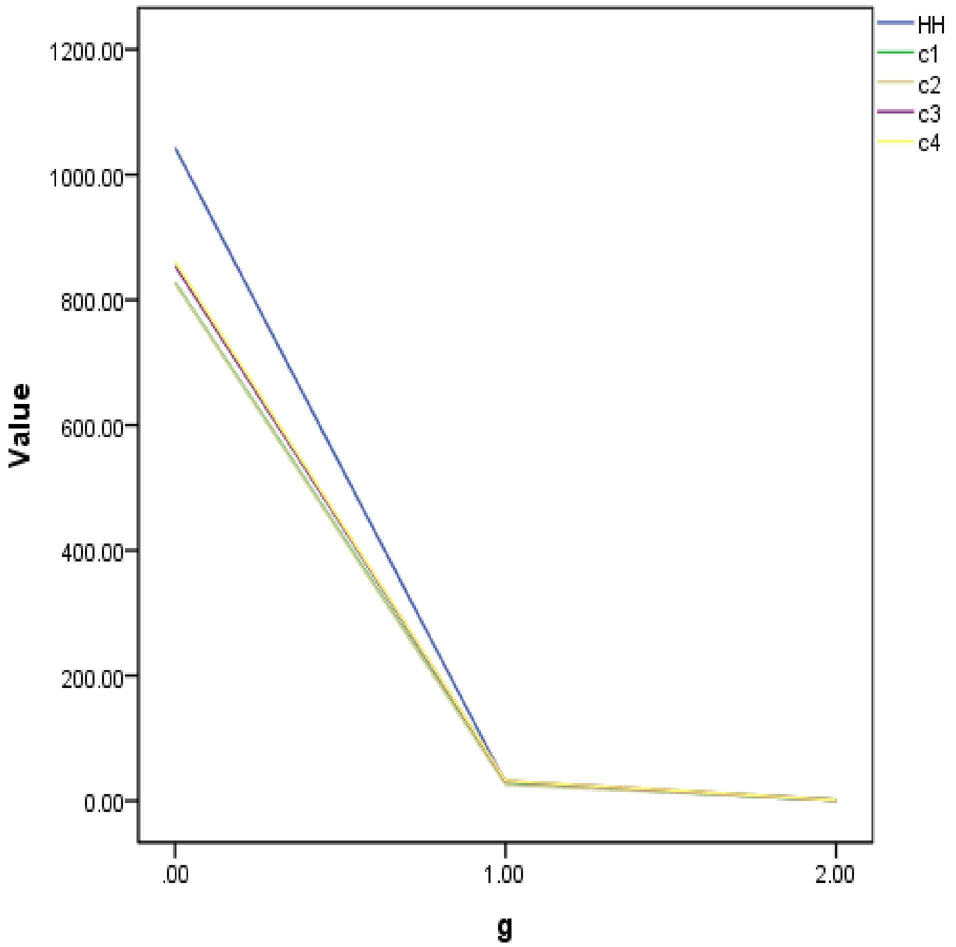
By these results, it is noticeable that there is no single estimator that is uniformly most efficient in the parameter space especially when correlation coefficient is weak. Even when correlation coefficient is high, there are points whereby other estimators perform equally well or even better than other estimators. This suggests the need to identify the conditions that bring about the change in estimators at varying levels of correlation coefficient.

5.4.3 Expected Mean squared Error of Alternative Linear Estimators as compared with Hansen- Hurwitz Estimator.

In this section, the super-population model described in (3.2.7) and hence, the expected mean squared error is utilized to generate the results for four study populations as displayed on tables 50, 51, 52 and 53 as shown in appendix A. Similarly, the graph of ξMSE is shown on figures 5, 6, 7, 8, 9 and 10 for populations I, II, III and IV respectively. The values of *g*, the super-population parameter used are usually, g = 0, 1 and 2. We consider the values of correlation coefficient ρ = 0, 0.1, 0.5, 0.9, 1 and the true population correlation coefficient estimated from the study population and defined as $\hat{\rho}$. To obtain definite values of ξMSE for comparison, it has been shown in Ekaette(2008) that the super-population parameter, α is minimized in the range [0,1]. To ensure non-negative variance (NNV), the values of α and β must be positive. In this study, we assume that the super-population parameters namely, α and β are equal to unity so that the per unit bias and hence, ξMSE can be determined under the model.

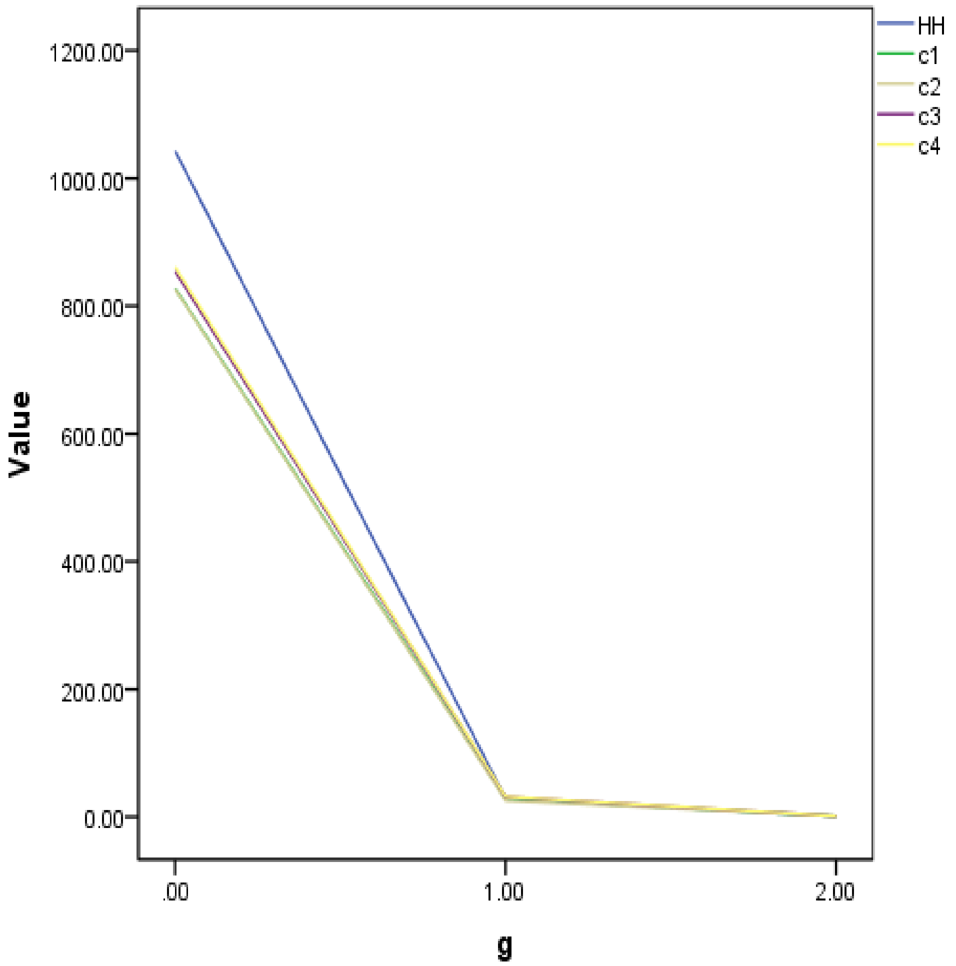
The results on tables 50 to 53 in appendix B below pertain to expected MSE (ξMSE) of alternative estimators as compared with HHE. The models describing the ξMSE for each estimator is shown along. However, the efficiency of the proposed estimators cannot be easily identified except when evaluated. For this reason, we consider the values of α=1 and β=1 which are the minimum integers that ensured the attainment non-negative variance . Thus, the results shown on figures 5 and 6 for populations I and II respectively; figures 7 and 8 for population III with ρ = -0.32 and ρ = 0.55 and also figures 9 and 10 for population IV with ρ = -0.77 and ρ = 0.91. Detailed description are shown below

Fig. 5: Graph of Expected MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population I



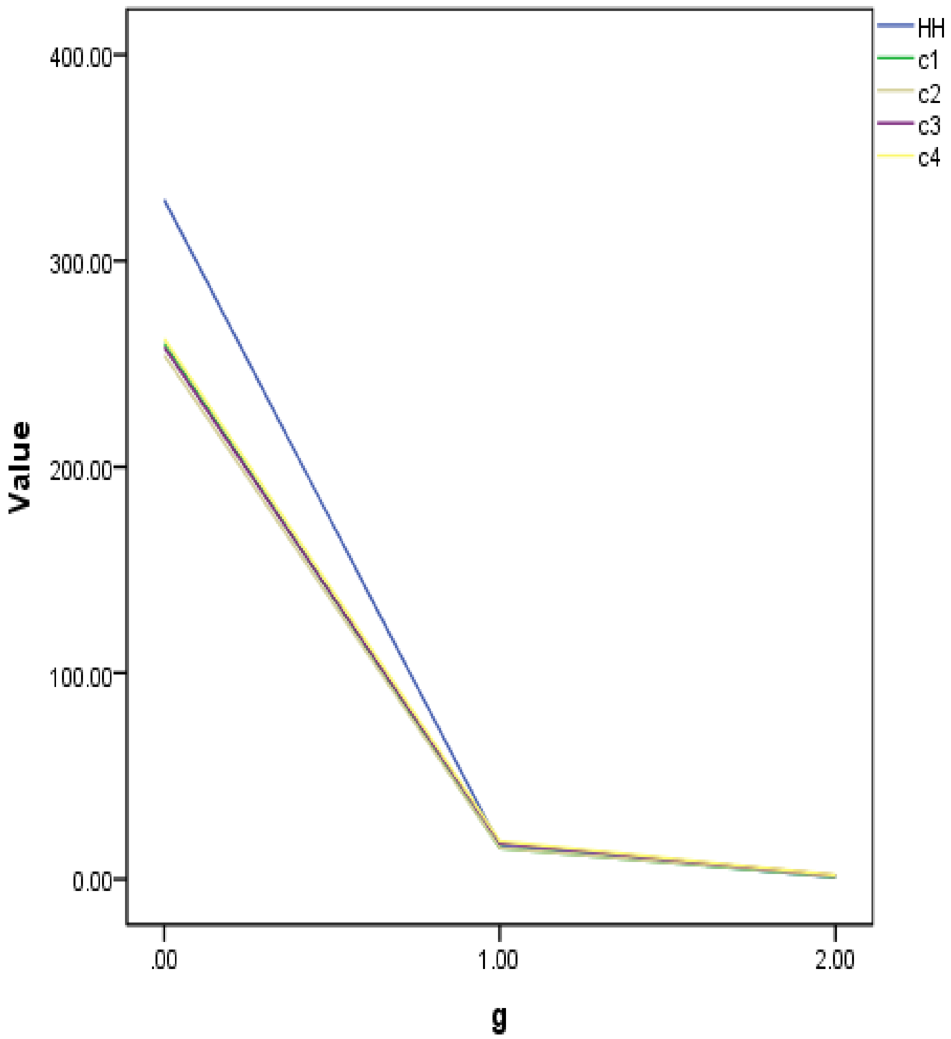
It is clear on figure 5 that the estimator $\hat{\tau}_{g,c=1}$ (with green line) and as shown on table 18 above, is the most efficient estimator with ξMSE of 131.2927 and 10.90 at g = 0 and g = 1 respectively. At g=2, the conventional estimator namely, Hansen and Hurwitz estimator has the lowest ξMSE of 0.9153. Thus, we conclude that the estimator defined by c = 1 is the best for population I at g = 0 and g = 1. When g=2, the conventional estimator becomes the best when compared with other estimators.

Fig. 6: Graph of Expected MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population II



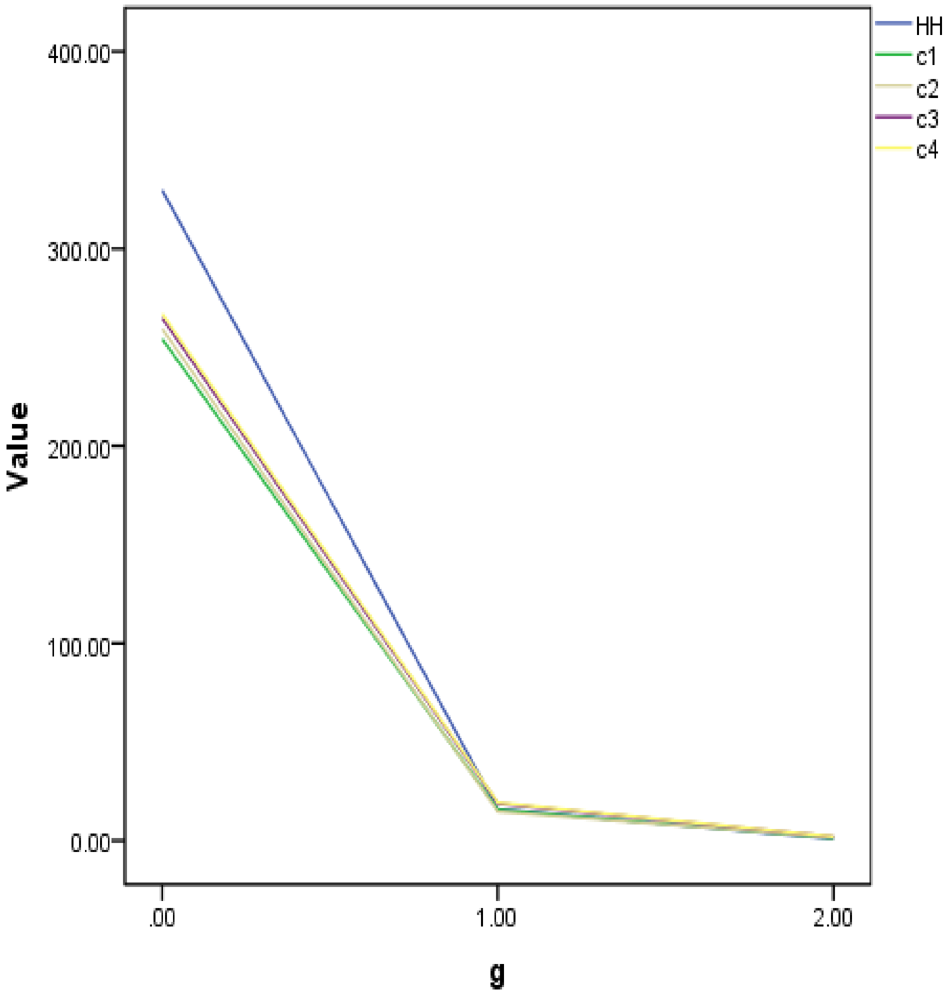
For population II, It can be seen on figure 6 that the estimator $\hat{\tau}_{g,c=2}$ (with brown line) and as shown on table 51 above, is the most efficient estimator with ξMSE of 826.5816 and 25.594 at g = 0 and g = 1 respectively. At g=2, the conventional estimator namely, Hansen and Hurwitz estimator has the lowest ξMSE of 0.96253. Again, we conclude that the estimator defined by c = 2 is the best for population II at g = 0 and g = 1. When g=2, the conventional estimator becomes the best when compared with other estimators.

Fig. 7: Graph of Expected MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population III



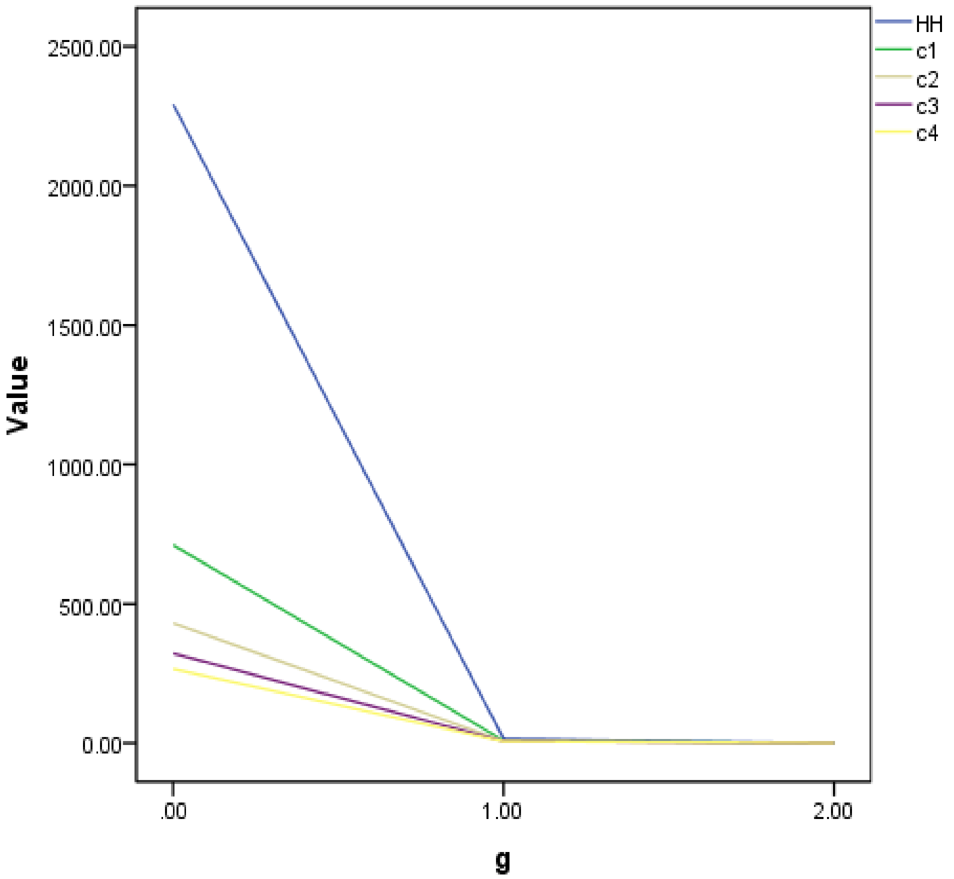
Population III has two sets ξMSE's computed at ρ=-0.32 and 0.51. Figure 7 shows the performance of the estimators at ρ=-0.32 while figure 8 shows the performance of the estimators at ρ=0.51 as contained on table 53. It can be seen on table 52 and also figure 7 that the estimator $\hat{\tau}_{g,c=2}$ (with brown line) is the most efficient estimator when ρ=-0.32 while the estimator defined by c = 1 as contained on figure 8 is relatively more efficient than all other estimators when ρ=0.51 with ξMSE of 254.301 and 15.737 at g = 0 and g = 1 respectively. At g=2, the conventional estimator namely, Hansen and Hurwitz estimator has the lowest ξMSE of 0.9273.

Fig. 8: Graph of Expected MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population III at rho=.55



For population IV, there are also two sets ξMSE's computed at ρ=-0.75 and 0.91. Figure 9 shows the performance of the estimators at ρ=-0.32 while figure 10 shows the performance of the estimators at ρ=0.51 as contained on table 53. It can be seen on table 53 and also figure 9 that the estimator $\hat{\tau}_{g,c=4}$ (with brown line) is the most efficient estimator when ρ=-0.77 and also at ρ=0.91 with ξMSE of 165.286 and 5.623 at g = 0 and g = 1 respectively. At g=2, the conventional estimator namely, Hansen and Hurwitz estimator has the lowest ξMSE of 0.2808. it is also noticeable that when negative correlation was encountered here, the estimator defined by c = 4 was optimum at all values of g with ξMSE of 266.310, 6.485 and 0.238 at g = 0, 1 and 2 respectively

Fig. 9: Graph of Expected MSE of alternative linear estimators as compared with Hansen and Hurwitz (HH) estimator for population III at rho=-0.77



The results on figures 5 to 9 exhibit the following characteristics:

a) The expected MSE, ξMSE is maximum at g = 0;
b) It attained minimum as g →2 in most cases;
c) The results in (a) and (b) above are consistent for all the four populations.

Table 18 below presents the summary of the results of moment estimators specified for the four study populations under the sampling design and super-population model. The results clearly shows the changing specification along moments in the correlation coefficient and also, that no single estimator is sufficient for all populations. Again, we can conveniently state that the estimators defined by c=1, c=2, c=2 and c=4 are best for populations I, II, III and IV respectively under the PPS sampling design. Similarly, under super-population model, the estimators defined by c=(1, 1,HHE), c=(2,2,HHE), c=(2,2,HHE) and also c = (4,4,HHE) for populations I, II, III and IV are best for g=0,1 and 2 respectively.

Table 18: Estimators defined by Moment in correlation coefficient in relation to the distribution of the study populations.

Population	Correlation		Design based estimator		Model based estimators (actual)			Model based estimators (Modified)		
	Actual	*Modified*	*Actual*	*Modified*	*g=0*	*g=1*	*g=2*	*g=0*	*g=1*	*g=2*
I	0.162	-	c =1	-	1	1	HHE	-	-	-
II	0.395	-	c =2	-	2	2	HHE	-	-	-
III	-0.321	0.55	c = 2	c = 2, 3	2	2	HHE	1	2	HHE
IV	-0.775	0.91	c = 4	c = 4	4	4	3	4	4	HHE

5.5 Approximate value of c at g = 0,1,2 for the study Populations.

Here, the expression for determining approximate value of c as given in 4.26 of chapter four is utilized and the study populations applied to obtain approximate values of the specification parameter, c. The results were earlier shown on table 13. Since the distributions consist of N values of p_i, N values of c are computed. However, our interest here is to obtain the lower and upper values of c determined by Min p_i and Max p_i.

From table 18 above, it is clear that the true value of c is in [0,2], [0,2], [0,3] and [0,13] for populations I, II, III and IV respectively when g=0. However, when g =1, c is a value in [0,2], [0,4], [0,3] and [0,6] for populations I, II, III and IV respectively. When g=2, c is a value in [0,1], [0,2], [0,2] and [0,8] for populations I, II, III and IV respectively.

By these results, we can conveniently state that the best value of c is determined when $g \to 2$, especially by the interval defined by Min p_i and Max p_i.

Again, for convenience, we assume that for the values of c =4 is adequate for all values of $c \geq 4$, since our interest is mainly in the first four moments described by c = 1, 2, 3 and 4.

Therefore, we can conclude that the specification parameter are c=1, c=2, c=2 and c=4 for populations I, II, III and IV respectively.

5.6 Relative Efficiency of alternative linear estimators as compared with Horvitz and Thompson estimator in πPS sampling design.

In this section, the relative efficiencies of the proposed estimators in PPSWOR sampling design otherwise, πPS design as compared with the conventional Horvitz and Thompson (1952) estimator are presented for populations I to IV as shown on tables 19 to 22 for sampling design while the expected MSE under super-population model are shown on tables 54 to 57 for populations I to IV respectively.

Tables 19 to 22 also show the estimates of population total, bias, variance and MSE for populations I to IV respectively.

Table 19: Estimates of population total, bias, variance and relative efficiency using conventional and alternative estimators in PPSWOR sampling design for population I

Estimators	$\hat{\tau}_{HT}$	$\hat{\tau}_{g,c=1}$	$\hat{\tau}_{g,c=2}$	$\hat{\tau}_{g,c=3}$	$\hat{\tau}_{g,c=4}$
Total	3023.07	2985.70	2985.83	2986.02	2986.05
Bias	0.00	-0.02	1.02	1.22	1.25
Variance	706919.29	681011.34	681079.26	681163.50	681143.82
RE	100.00	96.34	96.34	96.36	96.35

Table 20: Estimates of population total, bias, variance and relative efficiency using conventional and alternative estimators in PPSWOR sampling design for population II.

Estimators	$\hat{\tau}_{HT}$	$\hat{\tau}_{g,c=1}$	$\hat{\tau}_{g,c=2}$	$\hat{\tau}_{g,c=3}$	$\hat{\tau}_{g,c=4}$
Total	4700.59	4170.32	4138.16	4141.81	4145.53
Bias	0.00	-2.60	3.43	6.66	8.06
Variance	853381.41	588259.60	577833.43	580399.04	582315.44
RE	100.00	68.93	67.71	68.01	68.24

Table 21: Estimates of population total, bias, variance and relative efficiency using conventional and alternative estimators in PPSWOR sampling design for population III.

Estimators	$\hat{\tau}_{g,c=HT}$	$\hat{\tau}_{g,c=1}$	$\hat{\tau}_{g,c=2}$	$\hat{\tau}_{g,c=3}$	$\hat{\tau}_{g,c=4}$
Total	9805.67	8660.18	8602.48	8618.65	8628.06
Bias	0.00	-12.06	24.05	42.10	48.75
Variance	6925145.53	4751914.88	4626838.10	4642936.32	4655158.96
RE	100.00	68.62	66.81	67.04	67.22

Table 22: Estimates of population total, bias, variance and relative efficiency using conventional and alternative estimators in PPSWOR sampling design for population IV.

Estimators	$\hat{\tau}_{HT}$	$\hat{\tau}_{g,c=1}$	$\hat{\tau}_{g,c=2}$	$\hat{\tau}_{g,c=3}$	$\hat{\tau}_{g,c=4}$
Total	39720.98	17955.03	14236.65	12675.18	11859.44
Bias	0.00	-243.09	-273.84	-266.46	-243.02
Variance	117152937	19222566	11750358	9241840	8058239
RE	100.00	16.41	10.03	7.89	6.88

5.6.1 Relative Efficiencies of proposed estimators as compared with Horvitz and Thompson Estimator in PPSWOR design.

Considering table 19 above, it is clear that RE($\hat{\tau}_{HT} \backslash \hat{\tau}_{g,c=1}$) < RE($\hat{\tau}_{HT} \backslash \hat{\tau}_{g,c}$), c = 2,3,4 and also, RE($\hat{\tau}_{HT} \backslash \hat{\tau}_{g,c=1}$) < RE($\hat{\tau}_{HT} \backslash \hat{\tau}_{HT}$). Thus, the estimator defined by c=1 and c=2 are equally efficient with RE coefficient of 96.34% that minimizes the MSE as far as population I is concerned.

For population II, RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=2}$) < RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c}$), c = 1,2,3 and also RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=2}$) < RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$), hence the estimator defined by c=2 as shown on table 20 above with relative efficiency coefficient of 67.71% that minimizes the MSE. Similarly, in population III as shown on table 21 above, RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=2}$) < RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c}$), c = 1,3,4 and also RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=2}$) < RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$), so that the estimator defined by c=2 performed better than all other estimators in the class.

For population IV, RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=4}$) < RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c}$), c = 1,2,3 and also RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{g,c=4}$) < RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$), and so, it is clear that the estimator defined by c=4 is the best estimator in terms of minimizing MSE as shown on table 22 above.

5.6.2 Relative Efficiencies of proposed estimators as compared with Horvitz and Thompson Estimator under super-population model.

Under super-population model, the ξMSE for populations I, II, III and IV are shown on tables 54, 55, 56 and 57 in appendix B below. Their estimators are defined for populations I, II, II and IV respectively as follows:

Population I: for g = 0, 1, 2; c = 4, 1 and HTE;

Population II: for g = 0, 1, 2; c = 4, 1, 1;

Population II: for g = 0, 1, 2; c = 4, 1, 1;

Population II: for g = 0, 1, 2; c = 4, 4, 1.

We again observe the changing specification for the study populations, especially under the sampling design. These changing specification are of importance in this study as they relate with moment in correlation coefficient determined by the expectation of the c^{th} moment in the standardized variable and hence, the linear regression model.

5.7 Relative Efficiencies of proposed alternative linear estimators in pps sampling schemes under certain theoretical probability distributions.

In this section, the estimators are studied by utilizing theoretical distributions of the auxiliary information namely normal, uniform, chi-square and gamma distributions representing symmetric, rectangular and asymmetric distributions with the aim of determining whether the distributions of the variables have impact on the specification and hence, definition of an estimator for a target population.

Here, the auxiliary information is assumed to be known and the survey statistician obtains only the study variables for estimation. For this reason, we shall only simulate values of auxiliary variables and hence, the corresponding selection probabilities.

5.7.1 Relative efficiency of proposed estimators as compared with conventional estimator under the theoretical distributions for population I.

The results of the relative efficiencies of the alternative estimators in PPSWR sampling scheme as compared with the conventional estimator for both sampling design and super-population model are shown on tables 23 to 26 below.

Under normal distribution, the estimator defined by c = 4 is the best estimator with Relative efficiency coefficient of 7.6%. It is also clear from the result that the Rao's estimator corresponding with ρ=0 is as good as the estimator defined by c=4 as shown on table 23 below. Under chi-square distribution, the estimators defined by c = 1,2,3 and 4 are equally efficient with MSE far below that of the conventional estimator. Again, the Rao's estimator proves to be equally efficient as the defined estimators at ρ=0.162 as shown in table 24 below.

Table23: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under the theoretical Normal Distribution for population I

Rho	RE($\hat{\tau}_{HH} \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_1 \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_2 \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_3 \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_4 \backslash \hat{\tau}_{HH}$)
0.000	100.0	7.6	7.6	7.6	7.6
0.100	100.0	8.5	7.6	7.6	7.6
0.162	100.0	9.7	7.7	**7.6**	**7.6**
0.500	100.0	24.1	12.1	9.0	8.0
0.900	100.0	73.7	56.8	45.2	37.0
1.000	100.0	100.0	100.0	100.0	100.0

Table24: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under the Ch-squared distribution for population I

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	0.0	0.0	0.0	0.0
0.100	100.0	0.0	0.0	0.0	0.0
0.162	100.0	**0.0**	**0.0**	**0.0**	**0.0**
0.500	100.0	0.1	0.0	0.0	0.0
0.900	100.0	0.2	0.1	0.1	0.1
1.000	100.0	100.0	100.0	100.0	100.0

Table25: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under the uniform distribution for population I

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	100.0	100.0	100.0	100.0
0.100	100.0	100.0	100.0	100.0	100.0
0.162	100.0	100.0	100.0	100.0	100.0
0.500	100.0	100.0	100.0	100.0	100.0
0.900	100.0	100.0	100.0	100.0	100.0
1.000	100.0	100.0	100.0	100.0	100.0

For uniform distribution, all the estimators, that is, the conventional and all the alternative estimators performed the same in terms of MSE as shown on table 25 above. Under gamma distribution, c = 4 is the best estimator as shown on table 26 below. In the same manner, Rao's estimator is another competing estimator with equal RE with the estimator defined by c = 4.

Table26: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under the gamma distribution for population I

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	**0.1**	**0.1**	**0.1**	**0.1**
0.100	100.0	0.8	0.2	0.1	0.1
0.162	100.0	1.3	0.2	0.2	**0.1**
0.500	100.0	3.8	2.0	1.0	0.5
0.900	100.0	11.0	6.9	5.6	4.8
1.000	100.0	100.0	100.0	100.0	100.0

5.7.2 Relative efficiency of proposed estimators and the conventional estimator under certain theoretical distributions for population II.

We note from the analysis that under uniform distribution, the estimators are all the same. For this reason, we concentrate on the other distributions, namely, normal, chi-square and gamma distributions respectively. The results are presented in tables 27 to 29 below.

Under normal distribution, the estimator defined by c = 3 is the best having RE coefficient of 22% which is far less that all other estimators at ρ=0.395 as shown on table 27 below.

Estimators with c = 3 and 4 are best under chi-square distribution with RE coefficient of 0.1% which is again far less than all other estimators in terms of MSE. As shown on table 28 below. Under gamma distribution, the estimator defined by c = 4 is preferred to all others with relative mean square error of 1.5% as shown on table 29.

Table27: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under the theoretical Normal Distribution for population II

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	22.4	22.4	22.4	22.4
0.100	100.0	21.9	22.3	22.4	22.4
0.395	100.0	27.6	22.2	22.0	22.2
0.500	100.0	32.3	23.5	22.0	22.0
0.900	100.0	75.4	60.2	50.1	43.0
1.000	100.0	100.0	100.0	100.0	100.0

Table28: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under Chi-square Distribution for population II

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	0.1	0.1	0.1	0.1
0.100	100.0	0.2	0.1	0.1	0.1
0.395	100.0	0.6	0.2	0.1	0.1
0.500	100.0	0.8	0.4	0.2	0.1
0.900	100.0	2.4	1.5	1.2	1.0
1.000	100.0	100.0	100.0	100.0	100.0

Table29: Relative Efficiencies of proposed alternative estimators as compared with the Hansen and Hurwitz estimator under Gamma Distribution for population II

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	1.4	1.4	1.4	1.4
0.100	100.0	2.1	1.4	1.4	1.4
0.395	100.0	6.3	2.7	1.7	1.5
0.500	100.0	8.1	4.0	2.4	1.7
0.900	100.0	32.5	19.6	14.6	11.8
1.000	100.0	100.0	100.0	100.0	100.0

5.7.3 Relative efficiency of proposed estimators and the conventional estimator under certain theoretical distributions for population III.

The results of the relative efficiency as measured by the relative mean square error for population III are shown on tables 30, 31 and 32 for normal, chi-square and gamma distributions respectively. Again, it is clear that under normal distribution, the estimator defined by c= 4 is the best while those defined by c = 3and c =4 are the best estimators under chi-square distributions. Here, the Rao's estimator perfomed the same as the estimators earlier defined for the population. In the case of gamma distribution as shown on table 31, the estimator defined by c = 2 with RMSE = 28.45% for ρ=0.5 and c=1 with RMSE = 28.904% at ρ=-32 are the best for this population.

Table30: Relative Efficiencies of linear alternative estimators as compared with HHE for the theoretical Normal Distribution for population III

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	28.5	28.5	28.5	28.5
0.1	100.0	29.0	28.5	28.5	28.5
0.5	100.0	40.7	31.5	29.3	28.7
0.5	100.0	41.2	31.7	29.4	28.7
0.9	100.0	80.0	66.6	57.2	50.4
1.0	100.0	100.0	100.0	100.0	100.0

Table 31: Relative Efficiencies of linear alternative estimators as compared with HHE for the theoretical Chi square Distribution for population III

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	0.002	0.002	0.002	0.002
0.1	100.0	0.002	0.002	0.002	0.002
0.3	100.0	0.003	0.002	0.002	0.002
0.5	100.0	0.004	0.003	0.002	0.002
0.9	100.0	0.008	0.005	0.004	0.003
1.0	100.0	100.0	100.0	100.0	100.0

Table 32:Relative Efficiencies of linear alternative estimators as compared with HHE for the theoretical the theoretical Gamma Distribution for population III

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	30.986	30.986	30.986	30.986
0.1	100.0	29.030	30.723	30.959	30.984
0.3	100.0	28.904	29.000	30.184	30.711
0.5	100.0	32.787	28.450	28.681	29.508
0.9	100.0	67.776	51.944	42.982	37.473
1.0	100.0	100.0	100.0	100.0	100.0

5.7.4 Relative efficiency of proposed estimators and the conventional estimator under certain theoretical distributions for population IV.

We considered the results on tables 33, 34 and 35 below for normal, chi-squared and gamma distributions respectively. It can be observed here that under normal distribution, c=4 defined the best estimator with RMSE of 48.6% and 59.7% respectively at both |ρ|=0.775 and ρ=0.91.

These specifications are same for chi-square and gamma distributions as evidenced on tables 34 and 35 with RMSE of 1.9% and 2.9% for chi-square distribution and 7.1% and 9.7% for gamma distribution.

Table 33: Relative Efficiencies of linear alternative estimators as compared with HHE for the theoretical Normal Distribution for population IV

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	45.9	45.9	45.9	45.9
0.100	100.0	45.7	45.9	45.9	45.9
0.775	100.0	68.2	56.7	51.4	48.6
0.900	100.0	82.3	71.5	64.5	59.7
1.000	100.0	100.0	100.0	100.0	100.0

Table34: Relative Efficiencies of linear alternative estimators as compared with HHE for the theoretical Chi square Distribution for population IV

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	1.3	1.3	1.3	1.3
0.100	100.0	1.3	1.2	1.3	1.3
0.775	100.0	3.8	2.6	2.1	1.9
0.900	100.0	7.4	4.4	3.4	2.9
1.000	100.0	100.0	100.0	100.0	100.0

Table35: Relative Efficiencies of linear alternative estimators as compared with HHE for the theoretical Gamma Distribution for population IV

Rho	RE($\hat{\tau}_{HH} \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_1 \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_2 \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_3 \backslash \hat{\tau}_{HH}$)	RE($\hat{\tau}_4 \backslash \hat{\tau}_{HH}$)
0.000	100.0	4.4	4.4	4.4	4.4
0.100	100.0	5.3	4.5	4.4	4.4
0.775	100.0	12.6	9.0	7.8	7.1
0.900	100.0	22.8	14.2	11.2	9.7
1.000	100.0	100.0	100.0	100.0	100.0

Table 36 below presents the summary of the specification of estimators by utilizing theoretical distributions with the study variable which shows the varying estimators for varying populations and especially, as it relates with moment in ρ.

Table 36: Summary of estimators defined by c under the theoretical distributions for four study populations under PPS sampling design

Population	**Normal**	**Uniform**	**Chi-square**	**Gamma**
Population I	c=3,4	all estimators	c=1,2,3,4	c=1,2,3,4
Population II	c=3	all estimators	c=4	c=4
Population III	c=4	all estimators	c=3,4	c=2
Population IV	c=4	all estimators	c=4	c=4

It is clear from table 36 that estimators defined by higher moments c=3 or c = 4 are best for populations that are normally distributed. In the case of uniform distribution, all estimators performed equally. For chi-square and gamma distributions, there is a mixture of higher and lower moment and in cases where the study and measure of size variables are weakly correlated, all estimators performed equally the same.

5.8 Expected MSE of proposed and conventional estimators under certain theoretical distributions.

Detailed results are presented in appendix B below for the theoretical distributions and the study populations. For convenience, we present the summary of the estimators as shown on table 37 below for the study populations and distributions as adjudged by the performance of the ξMSE.

Table 37: specification of estimators by super-population model under theoretical distributions

Population/ Distribution	g	Population I	Population II	Population III	Population IV
Normal	0	c=4	c=1	c=2	c=2(4)
	1	c=4	c=1	c=1	c=1
	2	HHE	HHE	HHE	HHE
Uniform	0	c =1,2,3,4	c =1,2,3,4	c =1,2,3,4	c =1,2,3,4
	1	c =1,2,3,4	c =1,2,3,4	c =1,2,3,4	c =1,2,3,4
	2	HHE	HHE	HHE	HHE
Chi-square	0	c=4	c=1	c=1(1)	c=3(4)
	1	c=4	c=1	c=HHE(1)	c=1(2)
	2	HHE	HHE	HHE	HHE
Gamma	0	c=3	c=1	c=1(2)	c=4(4)
	1	c=3	c=1	c=1(1)	c=1(3)
	2	HHE	HHE	HHE	HHE

Note: *values in parenthesis represent the estimate at transformed ρ*

From table 37 above, it is clear that no single estimator can be said to be consistently efficient for all the study populations even under the super-population model. However, we cannot certainly conclude that the distribution of the population determines the specification of an estimator.

5.9 Comparison of estimators under Rao-Hartley Cochran scheme

In 4.62, the variance and hence MSE of the proposed estimator under RHC strategy was defined. In this section, the results of MSE is presented on tables 38 to 41 while those of relative efficiencies are shown on tables 42 to 45 for populations I to IV below respectively.

Table 38: Mean Squared Error of Alternative Linear Estimators under RHC scheme for Population I

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	2025.19	1155.92	1155.92	1155.92	1155.92
0.100	2025.19	1135.19	1152.61	1155.58	1155.89
0.162	2025.19	1135.62	1147.83	1154.48	1155.69
0.500	2025.19	1300.03	1152.69	1134.18	1139.79
0.900	2025.19	1831.36	1678.93	1558.58	1463.38
1.000	2025.19	2025.19	2025.19	2025.19	2025.19

Table 39: Mean Squared Error of Alternative Linear Estimators under RHC scheme for Population II

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	4964.55	1756.86	1756.86	1756.86	1756.86
0.370	4964.55	1508.65	1530.12	1651.84	1714.91
0.395	4964.55	1531.30	1512.08	1632.50	1703.08

0.500	4964.55	1682.64	1466.86	1542.92	1631.02
0.900	4964.55	3661.47	2902.39	2425.63	2110.76
1.000	4964.55	4964.55	4964.55	4964.55	4964.55

Table 40: Mean Squared Error of Alternative Linear Estimators under RHC scheme for Population III

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.1	111371.1	111371.1	111371.1	111371.1	111371.1
0.4	111371.1	93875.7	81430.8	72238.0	65257.2
0.5	111371.1	53597.2	43270.3	43251.5	45431.0
0.9	111371.1	45033.0	43827.8	47202.5	48915.7
1.0	111371.1	43902.1	48957.1	49758.5	49842.2
0.0	111371.1	49851.5	49851.5	49851.5	49851.5

Table 41: Mean Squared Error of Alternative Linear Estimators under RHC scheme for Population IV

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
1.0	1157246.0	1157246.0	1157246.0	1157246.0	1157246.0
0.9	1157246.0	566969.8	436895.8	379226.3	344495.3
0.8	1157246.0	392935.4	310935.9	265308.9	230137.3
0.5	1157246.0	276694.4	190937.4	148588.1	136151.5
0.1	1157246.0	142190.8	141803.8	145635.0	146083.4
0.0	1157246.0	146134.0	146134.0	146134.0	146134.0

The MSE of the alternative estimators as shown on tables 38 to 41above suggested that the estimators defined by c=1, c=2, c=3 and c=4 are best for populations I, II, III and IV with ρ = 0.162, ρ = 0.395, ρ = -0.32 or 0.55and ρ = -0.775or 0.91 respectively.

Table 42: Relative Efficiencies of Alternative Linear Estimators under RHC scheme for Population I

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	57.1	57.1	57.1	57.1
0.100	100.0	56.1	56.9	57.1	57.1
0.162	100.0	**56.1**	56.7	57.0	57.1
0.500	100.0	64.2	56.9	56.0	56.3
0.900	100.0	90.4	82.9	77.0	72.3
1.000	100.0	100.0	100.0	100.0	100.0

Table 43: Relative Efficiencies of Alternative Linear Estimators under RHC scheme for Population II

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
0.000	100.0	35.4	35.4	35.4	35.4
0.370	100.0	30.4	30.8	33.3	34.5
0.395	100.0	30.8	**30.5**	32.9	34.3
0.500	100.0	33.9	29.5	31.1	32.9
0.900	100.0	73.8	58.5	48.9	42.5
1.000	100.0	100.0	100.0	100.0	100.0

In terms of the RMSE, we investigate the specification of the estimator for each population which is our main focus in this study, it is observed that for population I, the estimator with c = 1is the best estimator with relative efficiency of 56.1%. As ρ increases to 0.5, the estimator defined by c = 3 would be preferred and the estimator defined by c = 4 would be the best when ρ→1.

Table 44: Relative Efficiencies of Alternative Linear Estimators under RHC scheme for Population III

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
1.000	100.0	100.0	100.0	100.0	100.0
0.9	100.0	84.3	73.1	64.9	58.6
0.5	100.0	48.1	38.9	**38.8**	40.8
0.3	100.0	40.4	**39.4**	42.4	43.9
0.1	100.0	39.4	44.0	44.7	44.8
0.0	100.0	44.8	44.8	44.8	44.8

Table 45: Relative Efficiencies of Alternative Linear Estimators under RHC scheme for Population IV

Rho	RE($\hat{\tau}_{HH}\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_1\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_2\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_3\backslash\hat{\tau}_{HH}$)	RE($\hat{\tau}_4\backslash\hat{\tau}_{HH}$)
1.000	100.0	100.0	100.0	100.0	100.0
0.9	100.0	49.0	37.8	32.8	**29.8**
0.8	100.0	34.0	26.9	22.9	**19.9**
0.5	100.0	23.9	16.5	12.8	11.8
0.1	100.0	12.3	12.3	12.6	12.6
0.0	100.0	12.6	12.6	12.6	12.6

In population II, the estimator defined by c = 2 is preferred at ρ=0.395 which is the true population correlation coefficient. The same specification would be appropriate for ρ=0.5 and as $\rho \to 1$, c= 4 would be preferred.

For population III, the true population correlation is -0.32. However under transformation, ρ=0.55. Considering ρ=.32, it is clear that the estimator with c = 2 is the best with relative efficiency of 39.5%. However, at ρ=.55 the estimator defined by c = 3 would be preferred as it has the lowest MSE expressed in terms of relative efficiency of 38.8%.

Finally, for population IV, the estimator defined by c = 4 is the best at ρ=0.775 with relative efficiency of 19.9%and ρ=0.91 and as $\rho \to 0$, the estimator with c = 1 would be preferred to all other estimators in the class.

All the estimators converge to HHE at ρ=1 and to Rao's estimator as ρ=0.

CHAPTER SIX

CONCLUSIONS

6.0 Introduction.

In this chapter, we summarize the main results of this thesis and suggest some areas of future research related to this study. Earlier, this thesis presented some literature including the popular works of Godambe(1955) and Basu(1971) that postulated the non existence of a uniformly most efficient homogenous estimator-theory which gave rise to finding alternative estimators. It was further observed that large-scale surveys have become complex in design and estimation for which the PPS sampling scheme is one of such designs that utilizes auxiliary information to enhance efficiency. However, most literature had emphasized the estimation of population parameter, say, total under the condition of positive correlation while the aspect of negative correlation is rarely addressed. The other issue of concern has been the assumption that each of the existing estimators in PPS sampling, both conventional and alternative estimators are efficient for all study populations irrespective of their distributional properties and the non-existence theory.

6.1 Main Results

In chapter three, we utilized the law of direct proportion to establish that the selection probabilities, p_i, is a realization of positive correlation between the study variables y and the measure of size x, which is an advancement on the classical ratio-estimator to obtain the Hansen-Hurwitz's estimator.

The selection probabilities, p_i's provided the normed-size measure for estimating population total under the PPSWR sampling scheme while the generalized selection probabilities $p^*_{i,g,c}$ provided a linear transformation that utilized the c^{th} (c=1,2,3,4) moment in correlation coefficient to develop a class of alternative linear estimators. We have shown that for efficiency, the relationship between the statistical properties namely coefficient of variation, skewness and kurtosis of the study variables and measure of size variables and correlation coefficient is expressed by $\rho^1 = \frac{CV_x}{CV_y} < 1$, $\rho^2 < 1$, $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1$; $\gamma_x \neq 0$ and $\rho^4 = \frac{K_y}{K_x} < 1$; $K_x \neq 0$.

When c = 1, we showed that $\rho^1 > \frac{CV_x}{CV_y} < 1$ along with the conditions namely $\rho^2 < 1$, $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1$and $\rho^4 = \frac{K_y}{K_x} < 1$ must hold true for the estimator defined by c = 1 to be utilized. This agrees with Cochran(1977) who showed that the ratio estimator is most efficient among other competing estimators when $\rho^1 > \frac{CV_x}{2CV_y} < 1$. However, this estimator can only be specified when $\rho \to 0$. This again agreed with the positions of Rao(1966), Bansal and singh(1985), Amahia et al(1989), Grewal(1999) among other scholars. We note here that this condition is only true for a linear estimator.

The study have also shown that when there is moment in ρ such that ρ takes a value $0.25<\rho<0.50$ or some neighbourhood and $\rho^2 \to 0$, then the estimator defined by c = 2 is best suitable for the target population. If there is further moment in ρ such that $\rho^3 = \frac{\gamma_y}{\gamma_x} < 1$ satisfying $0 < \rho^3 < 1$, then an estimator defined by c = 3 would be the best in term of MSE and relative efficiency. Empirical results have shown that this happens when $0.5<\rho<0.7$ and its neighbourhood. Similarly, when $\rho^4 = \frac{K_y}{K_x}<1$ and $0.7<\rho<0.99$, then the estimator defined by c = 4 is the best for the target population.

In situation where negatively correlated variables are encountered, direct transformation of measure of size variables could not provide the desired estimator. Thus, taking cognizance of the law of inverse proportion and further transformation from inverse to direct proportion by $p_i = \frac{1/x_i}{\sum_{i=1}^{N} 1/x_i} \Rightarrow p_i = \frac{z_i}{Z}$, the correlation structure changed from negative to positive correlation. By this transformation, a Modified Hansen-Hurwitz Estimator (MHHE) or Modified Horvitz-Thompson Estimator (MHTE) has been proposed for any observed case of negative correlation.

It is worth to note here that the MHHE or MHTE determined by $p_i = \frac{1/x_i}{\sum_{i=1}^{N} 1/x_i}$ possess the properties of harmonic mean which is mostly used when it is desirable to assign lower weights to higher values and higher weights to lower values. The selection probabilities and the derived generalized selection probabilities $p_{i,g,c}^{*}$ can be utilized in the class of alternative linear estimator by observing the conditions similar with those estimators involving positively correlated variables.

The interesting features of the developed estimators are that their bias, MSE's and ξMSE's coincide with the Rao's Estimator (RE) when $\rho=0$ and the conventional HHE or HTE in cases of PPSWR or πPS respectively when $\rho=1$. This provided boundaries for the linear estimators in PPS sampling scheme different from those defined by Sahoo(1995) estimator which had extended the boundaries but reduced the magnitude of negative correlation by restricting the estimators to instances of strong negative correlation.

The derived expression for determining approximate value of c is another useful means of defining an efficient estimator for a target population. Empirical evidence have shown that the optimum value of c lies between Min p_i and Max p_i.

The main aim of developing a general class of linear estimator is as a result of the fact of the non-existence of a uniformly most efficient estimator (UMEE) in the parameter space on one hand and the fact that no single estimator can be efficient for all populations and at all conditions. Thus, the class of alternative linear estimators defined by the generalized selection probabilities $p_{i,g}^{*} = \frac{1-\rho^c}{N} + \rho^c p_i$, c = 1,2,3,4 for $0<\rho<1$ provided the best estimators of population total for any target population.

The implication of the results above is that one estimator, say HHE, ACRE, RE among others cannot be said to be the best for all populations at all times. Thus different populations may have different estimators depending on their correlation coefficient and how it relates with the characteristics of the study populations.

The results of our empirical studies using sample of size two, that is, n =2 for the four study populations provided practical evidence of the behaviour of the developed class of linear estimators in PPS sampling schemes. Under the sampling design, we can conveniently infer that when $\rho=0$, the Rao's estimator would always be the best among other competing estimators. As ρ moves slightly from zero, Rao's estimator increased in bias and hence MSE and the anticipated MSE or ξMSE thereby suggesting the estimator defined by c = 1. As ρ moves closer to the 0.5, another estimator defined by c= 2 becomes the best among other competing estimators. Similarly, as ρ moves slightly away from 0.5 but not so strong, the estimator defined by c = 3 becomes the best. Furthermore, the estimator defined by c = 4 would be the best when it is clear that there is very strong correlation between the variables of study, especially when $\rho\rightarrow1$..

Certainly, the proposed estimators form a class of linear estimators bounded by Rao's estimator by the left and HHE by the right so that all other estimators defined by c = 1, 2, 3 and 4 are found within this class. Therefore, for a given population, the proposed linear estimators provide the best estimators for use in PPS sampling than utilizing the conventional estimator or a specified alternative estimator that are rigidly specified by fixed order of ρ.

The behaviour of the proposed estimators under the Rao-Hartley and Cochran scheme when n=5 is consistent with our earlier findings for n=2 in both PPS and πPS sampling schemes thereby suggests that increasing sample size would not change the estimators for the target populations. However, apart from uniform distribution for which all estimators are equal in performance, empirical evidence have shown that for theoretical populations that are normally distributed, estimators with c=3 or c = 4 performed better than other estimators in terms of MSE or ξMSE. However, for skewed distributions such as chi-squared and gamma distributions, estimators defined by c = 1 or 2 are best specified when $\rho\rightarrow0$ or somewhat moderate. However, as $\rho\rightarrow1$, estimators defined by c=3 or c=4 are best specified. Furthermore, the Grewals estimator is only best under super-population model than sampling design and utilizing this estimator would require transformation of c into $c^* =1/c$ as shown in chapter three.

Generally, the discussions above have shown that no single estimator is efficient for all populations and conditions and a rough idea of the magnitude of the correlation between the study and auxiliary variables and hence the size measures would provide insight into which estimator would be best for the target population. Secondly, the idea of the ratio of coefficients of variations, skewness and kurtosis as related with the correlation coefficient would help in the specification of estimators. Whereas the information of the target populations is not available to the survey statistician, this study have shown that among the estimators in the class defined by c=1,2,3 and 4 there is the one estimator that is best for estimating population total. Thus, in this era of information technology, it would be easier to identify such estimator when the suggested estimators are run simultaneously.

6.2 SUGGESTED AREA OF FUTURE RESEARCH

The identification of survey problems by the practicing survey statisticians have shown that survey data could also be subject to sampling and non-sampling errors. Secondly, the question of utilization of non-linear transformation for the selection probabilities as well as utilization of semi-parametric of non-parametric populations may arise. Thus this research could be extended to cover the areas of

- Non-sampling Error and possibly randomized response techniques (RRT);
- Non-linear transformation of selection probabilities;
- Utilization of Bayesian method to determining selection probabilities;
- Use of rank-correlation under the linear or non-linear transformations;
- Multi-stage PPS sampling as the case may be;
- Small domain estimation.

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APPENDIX A

STUDY POPULATIONS UTILIZED IN THIS STUDY

Table 46: Study Population I

S/No	x	y	P
1	3	11	0.395
2	4	7	0.395
3	5	9	0.395
4	8	8	0.395
5	12	8	0.395
6	11	9	0.395
7	8	8	0.395
8	9	12	0.395
9	11	10	0.395
10	10	9	0.395
11	8	8	0.395
12	9	14	0.395
13	7	12	0.395
14	8	10	0.395
15	8	10	0.395

S/No	x	Y	ρ
16	5	10	0.395
17	6	9	0.395
18	3	5	0.395
19	3	7	0.395
20	9	9	0.395
21	6	6	0.395
22	7	12	0.395
23	8	9	0.395
24	8	6	0.395
25	9	9	0.395
26	11	11	0.395
27	11	10	0.395
28	10	14	0.395
29	5	8	0.395
30	3	7	0.395

Table 47: Study Population II

S/No	X	y	P
1	41	36	0.162
2	43	47	0.162
3	54	41	0.162
4	39	47	0.162
5	49	47	0.162
6	45	45	0.162
7	41	32	0.162
8	33	37	0.162
9	37	40	0.162
10	41	41	0.162
11	47	37	0.162
12	39	48	0.162

Table 48:Study Population III

S/No	X	y	P
1	100	3	-0.32
2	88	8	-0.32
3	20	9	-0.32
4	17	11	-0.32
5	60	5	-0.32
6	77	9	-0.32
7	51	5	-0.32
8	69	4	-0.32
9	66	6	-0.32
10	77	9	-0.32
11	68	2	-0.32
12	36	4	-0.32
13	74	4	-0.32
14	33	5	-0.32
15	54	6	-0.32
16	55	6	-0.32
17	77	6	-0.32

Table 49: Population IV

S/No	x	y	P
1	6.8	20	-0.77
2	6.2	23	-0.77
3	5.5	38	-0.77
4	0.85	86	-0.77
5	0.71	92	-0.77
6	9	16	-0.77
7	1.4	81	-0.77
8	4.5	53	-0.77
9	3.8	42	-0.77
10	2.1	62	-0.77
11	4.85	39	-0.77
12	3.197	35	-0.77
13	0.443	87	-0.77
14	0.468	91	-0.77
15	0.59	84	-0.77
16	0.339	75	-0.77
17	0.161	54	-0.77
18	0.787	64	-0.77
19	0.069	26	-0.77
20	0.11	100	-0.77

APPENDIX B

ξMSE OF THE PROPOSED ALTERNATIVE ESTIMATORS FOR THE FOUR STUDY POPULATIONS

Table50: Relative Efficiency based on Expected MSE (ξMSE) of alternative estimators as compared with HHE for population I

g	Rho	Estimator	a			β^2	G	a			β^2	G	a			β^2
0	0.162	$\hat{\tau}_c$	134.319	a +	0	β^2	1	11	a +	0	β^2	**2**	**0.9153**	**a +**	**0**	β^2
		$\hat{\tau}_{g,c=1}$	131.281	a +	0.01167	β^2		10.890	a +	0.01167	β^2		0.94453	a +	0.01167	β^2
		$\hat{\tau}_{g,c=2}$	131.706	a +	0.01608	β^2		11.1282	a +	0.01608	β^2		0.9557	a +	0.01608	β^2
		$\hat{\tau}_{g,c=3}$	131.787	a +	0.01687	β^2		11.1432	a +	0.01687	β^2		0.95771	a +	0.01687	β^2
		$\hat{\tau}_{g,c=4}$	131.801	a +	0.017	β^2		11.1457	a +	0.017	β^2		0.95804	a +	0.017	β^2

Table51: Relative Efficiency based on Expected MSE (ξMSE) of alternative estimators as compared with HHE for population II

g	Rho	Estimator	a			β^2	G	a			β^2	G	A			β^2
0	0.395	$\hat{\tau}_c$	1042.98	a +	0	β^2	1	29	a +	0	β^2	2	**0.96253**	**a +**	**0**	β^2
		$\hat{\tau}_{g,c=1}$	**827.65**	**a +**	**0.03733**	β^2		28.5959	a +	0.03733	β^2		1.09179	a +	0.03733	β^2
		$\hat{\tau}_{g,c=2}$	826.513	a +	0.06859	β^2		25.5262	a +	0.06859	β^2		1.20495	a +	0.06859	β^2
		$\hat{\tau}_{g,c=3}$	854.467	a +	0.08467	β^2		31.5779	a +	0.08467	β^2		1.26148	a +	0.08467	β^2
		$\hat{\tau}_{g,c=4}$	861.352	a +	0.09176	β^2		32.0436	a +	0.09176	β^2		1.286	a +	0.09176	β^2

Table 52: Relative Efficiency based on Expected MSE (ξMSE) of alternative estimators as compared with HHE for population III

g	Rho	Estimator		A		β^2	G		a		β^2	G		a		β^2
0	0.5	$\hat{\tau}_c$	329.723	a +	0	β^2	1	16	a +	0	β^2	2	0.92725	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	260.038	a +	0.04894	β^2		14.9824	a +	0.04894	β^2		1.04657	a +	0.04894	β^2
		$\hat{\tau}_{g,c=2}$	254.2	a +	0.13684	β^2		14.8622	a +	0.13684	β^2		1.24678	a +	0.13684	β^2
		$\hat{\tau}_{g,c=3}$	257.917	a +	0.22283	β^2		17.2335	a +	0.22283	β^2		1.42167	a +	0.22283	β^2
		$\hat{\tau}_{g,c=4}$	261.977	a +	0.28608	β^2		18.0611	a +	0.28608	β^2		1.54175	a +	0.28608	β^2
0	0.32	$\hat{\tau}_c$	329.723	a +	0	β^2	1	16	a +	0	β^2	2	0.92725	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	254.197	a +	0.10403	β^2		15.6325	a +	0.10403	β^2		1.17519	a +	0.10403	β^2
		$\hat{\tau}_{g,c=2}$	259.189	a +	0.24371	β^2		14.5097	a +	0.24371	β^2		1.46198	a +	0.24371	β^2
		$\hat{\tau}_{g,c=3}$	264.557	a +	0.32309	β^2		18.5333	a +	0.32309	β^2		1.60951	a +	0.32309	β^2
		$\hat{\tau}_{g,c=4}$	266.811	a +	0.35449	β^2		18.927	a +	0.35449	β^2		1.66577	a +	0.35449	β^2

Table 53: Relative Efficiency based on Expected MSE (ξMSE) of alternative estimators as compared with HHE for population IV

g	Rho	Estimator	a			β²	G	a			β²	G	a.			β²
0	0.91	$\hat{\tau}_{c}$	2292.36	a +	0	β²	1	16	a +	0	β²	2	0.28083	a +	0	β²
		$\hat{\tau}_{g,c=1}$	711.025	a +	0.01273	β²		9.15609	a +	0.01273	β²		0.23119	a +	0.01273	β²
		$\hat{\tau}_{g,c=2}$	430.8	a +	0.03008	β²		7.5652	a +	0.03008	β²		0.20608	a +	0.03008	β²
		$\hat{\tau}_{g,c=3}$	322.159	a +	0.05026	β²		6.83628	a +	0.05026	β²		0.18485	a +	0.05026	β²
		$\hat{\tau}_{g,c=4}$	266.236	a +	0.07415	β²		6.41053	a +	0.07415	β²		0.16365	a +	0.07415	β²
0	0.775	$\hat{\tau}_{c}$	2292.36	a +	0	β²	1	16	a +	0	β²	2	0.28083	a +	0	β²
		$\hat{\tau}_{g,c=1}$	346.544	a +	0.04391	β²		7.00906	a +	0.04391	β²		0.19106	a +	0.04391	β²
		$\hat{\tau}_{g,c=2}$	222.85	a +	0.11534	β²		6.0349	a +	0.11534	β²		0.13087	a +	0.11534	β²
		$\hat{\tau}_{g,c=3}$	183.251	a +	0.22511	β²		5.59361	a +	0.22511	β²		0.0497	a +	0.22511	β²
		$\hat{\tau}_{g,c=4}$	164.902	a +	0.38348	β²		5.23992	a +	0.38348	β²		0.06507	a +	0.38348	β²

Table 54: Estimates of ξMSE using conventional and alternative estimators in PPSWOR sampling scheme for population I.

g	Rho	Estimator		A		β^2	g		A		β^2	G		a.		β^2
0	0.162	$\hat{\tau}_c$	757.7251	a+	0.000	β^2	1	61.04531	a+	0.000	β^2	2	4.995357	a+	0.000	β^2
		$\hat{\tau}_{g,c=1}$	720.7753	a+	0.038	β^2		59.61553	a+	0.038	β^2		5.010464	a+	0.038	β^2
		$\hat{\tau}_{g,c=2}$	719.4419	a+	0.051	β^2		59.7652	a+	0.051	β^2		5.045536	a+	0.051	β^2
		$\hat{\tau}_{g,c=3}$	719.346	a+	0.054	β^2		59.79987	a+	0.054	β^2		5.052158	a+	0.054	β^2
		$\hat{\tau}_{g,c=4}$	719.3336	a+	0.054	β^2		59.80576	a+	0.054	β^2		5.053256	a+	0.054	β^2

Table 55: Estimates of ξMSE using conventional and alternative estimators in PPSWOR sampling scheme for population II.

g	Rho	Estimator		A		β^2	g		A		β^2	G		a.		β^2
0	0.39	$\hat{\tau}_c$	22609.09	a+	0.000	β^2	1	504.0447	a+	0.000	β^2	2	13.96817	a+	0.000	β^2
		$\hat{\tau}_{g,c=1}$	13450.04	a+	0.429	β^2		398.2594	a+	0.429	β^2		13.74707	a+	0.429	β^2
		$\hat{\tau}_{g,c=2}$	12709.7	a+	0.721	β^2		404.3067	a+	0.721	β^2		14.66844	a+	0.721	β^2
		$\hat{\tau}_{g,c=3}$	12598.15	a+	0.859	β^2		410.831	a+	0.859	β^2		15.16344	a+	0.859	β^2
		$\hat{\tau}_{g,c=4}$	12577.8	a+	0.917	β^2		414.0036	a+	0.917	β^2		15.37947	a+	0.917	β^2

Table 56: Estimates of ξMSE using conventional and alternative estimators in PPSWOR sampling scheme for population III.

g	Rho	Estimator		A		β^2	g		A		β^2	G		a.		β^2
0	0.32	$\hat{\tau}_c$	3672.619	a+	0.00	β^2	1	153.996	a+	0.00	β^2	2	7.43251	a+	0.00	β^2
		$\hat{\tau}_{g,c=1}$	2299.603	a+	0.39	β^2		117.8413	a+	0.39	β^2		7.1915	a+	0.39	β^2
		$\hat{\tau}_{g,c=2}$	2171.411	a+	0.79	β^2		119.7481	a+	0.79	β^2		8.017956	a+	0.79	β^2
		$\hat{\tau}_{g,c=3}$	2152.597	a+	1.00	β^2		122.0586	a+	1.00	β^2		8.471023	a+	1.00	β^2
		$\hat{\tau}_{g,c=4}$	2149.15	a+	1.09	β^2		123.0378	a+	1.09	β^2		8.645418	a+	1.09	β^2

Table 57: Estimates of ξMSE using conventional and alternative estimators in PPSWOR sampling scheme for population IV.

G	Rho	Estimator		A		β^2	g		A		β^2	G		a.		β^2
0	0.77	$\hat{\tau}_c$	250723	a+	0.0	β^2	1	1039.224	a+	0.0	β^2	2	8.235249	a+	0.0	β^2
		$\hat{\tau}_{g,c=1}$	20067.68	a+	0.7	β^2		165.6597	a+	0.7	β^2		4.465283	a+	0.7	β^2
		$\hat{\tau}_{g,c=2}$	9544.162	a+	1.1	β^2		111.7766	a+	1.1	β^2		4.502901	a+	1.1	β^2
		$\hat{\tau}_{g,c=3}$	6518.028	a+	1.6	β^2		96.10361	a+	1.6	β^2		4.980096	a+	1.6	β^2
		$\hat{\tau}_{g,c=4}$	5191.764	a+	2.2	β^2		90.86967	a+	2.2	β^2		5.711622	a+	2.2	β^2

APPENDIX C

ξMSE OF THE PROPOSED ALTERNATIVE ESTIMATORS FOR THE THEORETICAL DISTRIBUTIONS OF THE MEASURE OF SIZE VARIABLES.

Table 58:Expected MSE of linear alternative estimators as compared with HHE for the theoretical Normal Distribution for population I

g	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	0.1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	132.3884	a +	0.018428	β^2		11.19912	a +	0.018428	β^2		0.962711	a +	0.018428	β^2
		$\hat{\tau}_{g,c=2}$	131.8173	a +	0.017061	β^2		11.14775	a +	0.017061	β^2		0.958255	a +	0.017061	β^2
		$\hat{\tau}_{g,c=3}$	131.8047	a +	0.017032	β^2		11.14629	a +	0.017032	β^2		0.958115	a +	0.017032	β^2
		$\hat{\tau}_{g,c=4}$	131.8039	a +	0.01703	β^2		11.14618	a +	0.01703	β^2		0.958104	a +	0.01703	β^2
0	0.162	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	133.267	a +	0.020548	β^2		11.27445	a +	0.020548	β^2		0.969096	a +	0.020548	β^2
		$\hat{\tau}_{g,c=2}$	131.8603	a +	0.017162	β^2		11.15207	a +	0.017162	β^2		0.958647	a +	0.017162	β^2
		$\hat{\tau}_{g,c=3}$	131.8084	a +	0.01704	β^2		11.14674	a +	0.01704	β^2		0.95816	a +	0.01704	β^2
		$\hat{\tau}_{g,c=4}$	131.8045	a +	0.017031	β^2		11.14625	a +	0.017031	β^2		0.958112	a +	0.017031	β^2

0	0.5	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	146.3695	a +	0.053071	β^2		12.36458	a +	0.053071	β^2		1.05984	a +	0.053071	β^2
		$\hat{\tau}_{g,c=2}$	135.2385	a +	0.025334	β^2		11.44082	a +	0.025334	β^2		0.983076	a +	0.025334	β^2
		$\hat{\tau}_{g,c=3}$	132.6938	a +	0.019164	β^2		11.22548	a +	0.019164	β^2		0.964953	a +	0.019164	β^2
		$\hat{\tau}_{g,c=4}$	132.0511	a +	0.017617	β^2		11.16953	a +	0.017617	β^2		0.960173	a +	0.017617	β^2
0	0.9	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	193.7988	a +	0.184093	β^2		16.23106	a +	0.184093	β^2		1.37615	a +	0.184093	β^2
		$\hat{\tau}_{g,c=2}$	178.0379	a +	0.138435	β^2		14.95324	a +	0.138435	β^2		1.272178	a +	0.138435	β^2
		$\hat{\tau}_{g,c=3}$	166.9784	a +	0.107597	β^2		14.05313	a +	0.107597	β^2		1.198654	a +	0.107597	β^2
		$\hat{\tau}_{g,c=4}$	158.9573	a +	0.085902	β^2		13.39801	a +	0.085902	β^2		1.144957	a +	0.085902	β^2
0	1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	217.3183	a +	0.255567	β^2		18.12977	a +	0.255567	β^2		1.529953	a +	0.255567	β^2
		$\hat{\tau}_{g,c=2}$	217.3183	a +	0.255567	β^2		18.12977	a +	0.255567	β^2		1.529953	a +	0.255567	β^2
		$\hat{\tau}_{g,c=3}$	217.3183	a +	0.255567	β^2		18.12977	a +	0.255567	β^2		1.529953	a +	0.255567	β^2
		$\hat{\tau}_{g,c=4}$	217.3183	a +	0.255567	β^2		18.12977	a +	0.255567	β^2		1.529953	a +	0.255567	β^2

Table59:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical chi square Distribution for population I

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	0.1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	135.6999	a +	0.034894	β^2		11.57401	a +	0.034894	β^2		1.003628	a +	0.034894	β^2
		$\hat{\tau}_{g,c=2}$	131.9305	a +	0.01807	β^2		11.16635	a +	0.01807	β^2		0.960684	a +	0.01807	β^2
		$\hat{\tau}_{g,c=3}$	131.8139	a +	0.017127	β^2		11.14796	a +	0.017127	β^2		0.958342	a +	0.017127	β^2
		$\hat{\tau}_{g,c=4}$	131.8048	a +	0.01704	β^2		11.14634	a +	0.01704	β^2		0.958127	a +	0.01704	β^2
0	0.162	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	141.1793	a +	0.054889	β^2		12.10459	a +	0.054889	β^2		1.055195	a +	0.054889	β^2
		$\hat{\tau}_{g,c=2}$	132.2603	a +	0.020108	β^2		11.20976	a +	0.020108	β^2		0.965799	a +	0.020108	β^2
		$\hat{\tau}_{g,c=3}$	131.8506	a +	0.017452	β^2		11.15414	a +	0.017452	β^2		0.959148	a +	0.017452	β^2
		$\hat{\tau}_{g,c=4}$	131.8107	a +	0.017096	β^2		11.1474	a +	0.017096	β^2		0.958267	a +	0.017096	β^2

0	0.5	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	242.7276	a +	0.410393	β^2		21.32146	a +	0.410393	β^2		1.903468	a +	0.410393	β^2
		$\hat{\tau}_{g,c=2}$	153.6932	a +	0.097924	β^2		13.27229	a +	0.097924	β^2		1.165272	a +	0.097924	β^2
		$\hat{\tau}_{g,c=3}$	137.6106	a +	0.042065	β^2		11.76194	a +	0.042065	β^2		1.022119	a +	0.042065	β^2
		$\hat{\tau}_{g,c=4}$	133.5498	a +	0.026226	β^2		11.35419	a +	0.026226	β^2		0.98136	a +	0.026226	β^2
0	0.9	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	2980.232	a +	13.59931	β^2		261.2076	a +	13.59931	β^2		23.21023	a +	13.59931	β^2
		$\hat{\tau}_{g,c=2}$	1038.571	a +	3.896042	β^2		91.90541	a +	3.896042	β^2		8.257873	a +	3.896042	β^2
		$\hat{\tau}_{g,c=3}$	591.4351	a +	1.841051	β^2		52.38213	a +	1.841051	β^2		4.712762	a +	1.841051	β^2
		$\hat{\tau}_{g,c=4}$	411.4918	a +	1.072098	β^2		36.39137	a +	1.072098	β^2		3.269943	a +	1.072098	β^2
2	1	$\hat{\tau}_c$	0.915304	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	2078113	a +	2064369	β^2		25797900	a +	2064369	β^2		2078113	a +	2064369	β^2
		$\hat{\tau}_{g,c=2}$	2078113	a +	2064369	β^2		25797900	a +	2064369	β^2		2078113	a +	2064369	β^2
		$\hat{\tau}_{g,c=3}$	2078113	a +	2064369	β^2		25797900	a +	2064369	β^2		2078113	a +	2064369	β^2
		$\hat{\tau}_{g,c=4}$	2078113	a +	2064369	β^2		25797900	a +	2064369	β^2		2078113	a +	2064369	β^2

Table60:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Uniform Distribution for population I

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	0.1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	0.162	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	0.5	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2

0	0.9	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2

Table61:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Gamma Distribution for population I

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=2}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=3}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
		$\hat{\tau}_{g,c=4}$	131.8038	a +	0.01703	β^2		11.14617	a +	0.01703	β^2		0.958103	a +	0.01703	β^2
0	0.1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	137.1507	a +	0.024525	β^2		11.52525	a +	0.024525	β^2		0.98454	a +	0.024525	β^2
		$\hat{\tau}_{g,c=2}$	131.808	a +	0.016693	β^2		11.14059	a +	0.016693	β^2		0.957157	a +	0.016693	β^2
		$\hat{\tau}_{g,c=3}$	131.7977	a +	0.016981	β^2		11.14507	a +	0.016981	β^2		0.957964	a +	0.016981	β^2
		$\hat{\tau}_{g,c=4}$	131.8032	a +	0.017025	β^2		11.14605	a +	0.017025	β^2		0.958089	a +	0.017025	β^2
0	0.162	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	145.307	a +	0.038028	β^2		12.15674	a +	0.038028	β^2		1.033699	a +	0.038028	β^2
		$\hat{\tau}_{g,c=2}$	132.1061	a +	0.016787	β^2		11.15541	a +	0.016787	β^2		0.957588	a +	0.016787	β^2
		$\hat{\tau}_{g,c=3}$	131.7878	a +	0.016847	β^2		11.14234	a +	0.016847	β^2		0.957581	a +	0.016847	β^2
		$\hat{\tau}_{g,c=4}$	131.7994	a +	0.016996	β^2		11.1454	a +	0.016996	β^2		0.958006	a +	0.016996	β^2
0	0.5	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	274.4888	a +	0.279584	β^2		22.45316	a +	0.279584	β^2		1.8628	a +	0.279584	β^2
		$\hat{\tau}_{g,c=2}$	162.8866	a +	0.067723	β^2		13.54195	a +	0.067723	β^2		1.143754	a +	0.067723	β^2
		$\hat{\tau}_{g,c=3}$	140.0335	a +	0.029269	β^2		11.7467	a +	0.029269	β^2		1.001621	a +	0.029269	β^2
		$\hat{\tau}_{g,c=4}$	133.8967	a +	0.019296	β^2		11.28071	a +	0.019296	β^2		0.966165	a +	0.019296	β^2

0	0.9	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	2642.948	a +	8.162984	β^2		214.5193	a +	8.162984	β^2		17.59732	a +	8.162984	β^2
		$\hat{\tau}_{g,c=2}$	1094.886	a +	2.556025	β^2		88.56306	a +	2.556025	β^2		7.247816	a +	2.556025	β^2
		$\hat{\tau}_{g,c=3}$	658.2176	a +	1.233234	β^2		53.28872	a +	1.233234	β^2		4.368034	a +	1.233234	β^2
		$\hat{\tau}_{g,c=4}$	466.8206	a +	0.723698	β^2		37.88626	a +	0.723698	β^2		3.114942	a +	0.723698	β^2
0	1	$\hat{\tau}_c$	134.3193	a +	0	β^2	1	11	a +	0	β^2	2	0.915304	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	45003.38	a +	216.4849	β^2		3642.2	a +	216.4849	β^2		297.0335	a +	216.4849	β^2
		$\hat{\tau}_{g,c=2}$	45003.38	a +	216.4849	β^2		3642.2	a +	216.4849	β^2		297.0335	a +	216.4849	β^2
		$\hat{\tau}_{g,c=3}$	45003.38	a +	216.4849	β^2		3642.2	a +	216.4849	β^2		297.0335	a +	216.4849	β^2
		$\hat{\tau}_{g,c=4}$	45003.38	a +	216.4849	β^2		3642.2	a +	216.4849	β^2		297.0335	a +	216.4849	β^2

Table62:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Normal Distribution for population II

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	1051.816	a +	0	β^2	1	29	a +	0	β^2	2	0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	865.8536	a +	0.108431	β^2		32.74248	a +	0.108431	β^2		1.343367	a +	0.108431	β^2
		$\hat{\tau}_{g,c=2}$	865.8536	a +	0.108431	β^2		32.74248	a +	0.108431	β^2		1.343367	a +	0.108431	β^2
		$\hat{\tau}_{g,c=3}$	865.8536	a +	0.108431	β^2		32.74248	a +	0.108431	β^2		1.343367	a +	0.108431	β^2
		$\hat{\tau}_{g,c=4}$	865.8536	a +	0.108431	β^2		32.74248	a +	0.108431	β^2		1.343367	a +	0.108431	β^2
0	0.1	$\hat{\tau}_c$	1051.816	a +	0	β^2	1	29	a +	0	β^2	2	0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	845.8217	a +	0.087788	β^2		31.35343	a +	0.087788	β^2		1.269936	a +	0.087788	β^2
		$\hat{\tau}_{g,c=2}$	863.5582	a +	0.106219	β^2		32.59258	a +	0.106219	β^2		1.33555	a +	0.106219	β^2
		$\hat{\tau}_{g,c=3}$	865.621	a +	0.108208	β^2		32.72737	a +	0.108208	β^2		1.342581	a +	0.108208	β^2
		$\hat{\tau}_{g,c=4}$	865.8303	a +	0.108408	β^2		32.74097	a +	0.108408	β^2		1.343289	a +	0.108408	β^2
0	0.395	$\hat{\tau}_c$	1051.816	a +	0	β^2	1	29	a +	0	β^2	2	0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	824.0128	a +	0.042208	β^2		28.52622	a +	0.042208	β^2		1.105386	a +	0.042208	β^2
		$\hat{\tau}_{g,c=2}$	837.3864	a +	0.07753	β^2		30.67661	a +	0.07753	β^2		1.233096	a +	0.07753	β^2
		$\hat{\tau}_{g,c=3}$	852.7463	a +	0.095335	β^2		31.85808	a +	0.095335	β^2		1.296896	a +	0.095335	β^2
		$\hat{\tau}_{g,c=4}$	860.3805	a +	0.103107	β^2		32.38195	a +	0.103107	β^2		1.324527	a +	0.103107	β^2
0	0.5	$\hat{\tau}_c$	1051.816	a +	0	β^2	1	29	a +	0	β^2	2	0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	830.7452	a +	0.030244	β^2		27.94087	a +	0.030244	β^2		1.062463	a +	0.030244	β^2
		$\hat{\tau}_{g,c=2}$	827.6869	a +	0.06214	β^2		29.69287	a +	0.06214	β^2		1.17751	a +	0.06214	β^2
		$\hat{\tau}_{g,c=3}$	841.8129	a +	0.083104	β^2		31.04279	a +	0.083104	β^2		1.253137	a +	0.083104	β^2
		$\hat{\tau}_{g,c=4}$	852.5789	a +	0.095158	β^2		31.84625	a +	0.095158	β^2		1.296269	a +	0.095158	β^2

0	0.9	$\hat{\tau}_c$	1051.816	a +	0	β^2	1	29	a +	0	β^2	2	0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	967.1424	a +	0.001732	β^2		28.07782	a +	0.001732	β^2		0.966477	a +	0.001732	β^2
		$\hat{\tau}_{g,c=2}$	914.6792	a +	0.005591	β^2		27.63903	a +	0.005591	β^2		0.978196	a +	0.005591	β^2
		$\hat{\tau}_{g,c=3}$	880.8265	a +	0.010472	β^2		27.48049	a +	0.010472	β^2		0.993993	a +	0.010472	β^2
		$\hat{\tau}_{g,c=4}$	858.5772	a +	0.015843	β^2		27.49419	a +	0.015843	β^2		1.01207	a +	0.015843	β^2
0	1	$\hat{\tau}_c$	1051.816	a +	0	β^2	1	29	a +	0	β^2	2	0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	1051.816	a +	0	β^2		29	a +	0	β^2		0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	1051.816	a +	0	β^2		29	a +	0	β^2		0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	1051.816	a +	0	β^2		29	a +	0	β^2		0.96206	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	1051.816	a +	0	β^2		29	a +	0	β^2		0.96206	a +	0	β^2

Table63:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical chi square Distribution for population II

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	36308.52	a +	0	β^2	1	29	a +	0	β^2	2	0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	829.6751	a +	0.908103	β^2		63.92168	a +	0.908103	β^2		5.783388	a +	0.908103	β^2
		$\hat{\tau}_{g,c=2}$	829.6751	a +	0.908103	β^2		63.92168	a +	0.908103	β^2		5.783388	a +	0.908103	β^2
		$\hat{\tau}_{g,c=3}$	829.6751	a +	0.908103	β^2		63.92168	a +	0.908103	β^2		5.783388	a +	0.908103	β^2
		$\hat{\tau}_{g,c=4}$	829.6751	a +	0.908103	β^2		63.92168	a +	0.908103	β^2		5.783388	a +	0.908103	β^2
0	0.1	$\hat{\tau}_c$	36308.52	a +	0	β^2	1	29	a +	0	β^2	2	0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	664.2674	a +	0.526584	β^2		47.1417	a +	0.526584	β^2		4.117038	a +	0.526584	β^2
		$\hat{\tau}_{g,c=2}$	808.5584	a +	0.856564	β^2		61.79399	a +	0.856564	β^2		5.570925	a +	0.856564	β^2
		$\hat{\tau}_{g,c=3}$	827.5057	a +	0.902771	β^2		63.70318	a +	0.902771	β^2		5.761554	a +	0.902771	β^2
		$\hat{\tau}_{g,c=4}$	829.4576	a +	0.907568	β^2		63.89977	a +	0.907568	β^2		5.781199	a +	0.907568	β^2
0	0.395	$\hat{\tau}_c$	36308.52	a +	0	β^2	1	29	a +	0	β^2	2	0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	481.2405	a +	0.141577	β^2		25.77527	a +	0.141577	β^2		2.022115	a +	0.141577	β^2
		$\hat{\tau}_{g,c=2}$	603.9834	a +	0.400059	β^2		40.85706	a +	0.400059	β^2		3.498724	a +	0.400059	β^2
		$\hat{\tau}_{g,c=3}$	717.119	a +	0.642909	β^2		52.54465	a +	0.642909	β^2		4.651189	a +	0.642909	β^2
		$\hat{\tau}_{g,c=4}$	780.3777	a +	0.789056	β^2		58.95113	a +	0.789056	β^2		5.287552	a +	0.789056	β^2
0	0.5	$\hat{\tau}_c$	36308.52	a +	0	β^2	1	29	a +	0	β^2	2	0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	473.1934	a +	0.092116	β^2		22.29896	a +	0.092116	β^2		1.675077	a +	0.092116	β^2
		$\hat{\tau}_{g,c=2}$	534.9548	a +	0.260827	β^2		33.237	a +	0.260827	β^2		2.752832	a +	0.260827	β^2
		$\hat{\tau}_{g,c=3}$	635.1936	a +	0.464789	β^2		44.13358	a +	0.464789	β^2		3.820684	a +	0.464789	β^2
		$\hat{\tau}_{g,c=4}$	715.7946	a +	0.63993	β^2		52.40998	a +	0.63993	β^2		4.637846	a +	0.63993	β^2

0	0.9	$\hat{\tau}_c$	36308.52	a +	0	β^2	1	29	a +	0	β^2	2	0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	1100.386	a +	0.009496	β^2		17.73342	a +	0.009496	β^2		0.991647	a +	0.009496	β^2
		$\hat{\tau}_{g,c=2}$	704.5309	a +	0.020492	β^2		17.47576	a +	0.020492	β^2		1.088707	a +	0.020492	β^2
		$\hat{\tau}_{g,c=3}$	574.6131	a +	0.032946	β^2		18.0484	a +	0.032946	β^2		1.199124	a +	0.032946	β^2
		$\hat{\tau}_{g,c=4}$	516.352	a +	0.047071	β^2		18.98861	a +	0.047071	β^2		1.320375	a +	0.047071	β^2
0	1	$\hat{\tau}_c$	36308.52	a +	0	β^2	1	29	a +	0	β^2	2	0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	36308.52	a +	0	β^2		29	a +	0	β^2		0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	36308.52	a +	0	β^2		29	a +	0	β^2		0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	36308.52	a +	0	β^2		29	a +	0	β^2		0.921861	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	36308.52	a +	0	β^2		29	a +	0	β^2		0.921861	a +	0	β^2

Table64:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Gamma Distribution for population II

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	3166.908	a +	0	β^2	1	29	a +	0	β^2	2	0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	845.3701	a +	0.646209	β^2		50.66768	a +	0.646209	β^2		3.654303	a +	0.646209	β^2
		$\hat{\tau}_{g,c=2}$	845.3701	a +	0.646209	β^2		50.66768	a +	0.646209	β^2		3.654303	a +	0.646209	β^2
		$\hat{\tau}_{g,c=3}$	845.3701	a +	0.646209	β^2		50.66768	a +	0.646209	β^2		3.654303	a +	0.646209	β^2
		$\hat{\tau}_{g,c=4}$	845.3701	a +	0.646209	β^2		50.66768	a +	0.646209	β^2		3.654303	a +	0.646209	β^2
0	0.1	$\hat{\tau}_c$	3166.908	a +	0	β^2	1	29	a +	0	β^2	2	0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	738.2882	a +	0.430939	β^2		41.03276	a +	0.430939	β^2		2.864421	a +	0.430939	β^2
		$\hat{\tau}_{g,c=2}$	832.1993	a +	0.619401	β^2		49.51382	a +	0.619401	β^2		3.559611	a +	0.619401	β^2
		$\hat{\tau}_{g,c=3}$	844.024	a +	0.643463	β^2		50.55002	a +	0.643463	β^2		3.644645	a +	0.643463	β^2
		$\hat{\tau}_{g,c=4}$	845.2352	a +	0.645933	β^2		50.65589	a +	0.645933	β^2		3.653336	a +	0.645933	β^2
0	0.395	$\hat{\tau}_c$	3166.908	a +	0	β^2	1	29	a +	0	β^2	2	0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	624.5125	a +	0.148824	β^2		26.6606	a +	0.148824	β^2		1.670031	a +	0.148824	β^2
		$\hat{\tau}_{g,c=2}$	697.506	a +	0.34845	β^2		37.07197	a +	0.34845	β^2		2.539546	a +	0.34845	β^2
		$\hat{\tau}_{g,c=3}$	773.4345	a +	0.501154	β^2		44.27541	a +	0.501154	β^2		3.130129	a +	0.501154	β^2
		$\hat{\tau}_{g,c=4}$	814.3856	a +	0.583344	β^2		47.94323	a +	0.583344	β^2		3.43078	a +	0.583344	β^2
0	0.5	$\hat{\tau}_c$	3166.908	a +	0	β^2	1	29	a +	0	β^2	2	0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	636.0752	a +	0.102242	β^2		24.1271	a +	0.102242	β^2		1.441977	a +	0.102242	β^2
		$\hat{\tau}_{g,c=2}$	651.5555	a +	0.247725	β^2		31.9719	a +	0.247725	β^2		2.118866	a +	0.247725	β^2
		$\hat{\tau}_{g,c=3}$	718.6735	a +	0.391549	β^2		39.16339	a +	0.391549	β^2		2.711176	a +	0.391549	β^2
		$\hat{\tau}_{g,c=4}$	772.5635	a +	0.499413	β^2		44.19631	a +	0.499413	β^2		3.123647	a +	0.499413	β^2

0	0.9	$\hat{\tau}_c$	3166.908	a +	0	β^2	1	29	a +	0	β^2	2	0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	1356.109	a +	0.009721	β^2		22.63467	a +	0.009721	β^2		0.973363	a +	0.009721	β^2
		$\hat{\tau}_{g,c=2}$	964.2044	a +	0.023587	β^2		21.43753	a +	0.023587	β^2		1.037457	a +	0.023587	β^2
		$\hat{\tau}_{g,c=3}$	801.4107	a +	0.038699	β^2		21.40077	a +	0.038699	β^2		1.114033	a +	0.038699	β^2
		$\hat{\tau}_{g,c=4}$	718.0063	a +	0.054962	β^2		21.86685	a +	0.054962	β^2		1.198751	a +	0.054962	β^2
0	1	$\hat{\tau}_c$	3166.908	a +	0	β^2	1	29	a +	0	β^2	2	0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	3166.908	a +	0	β^2		29	a +	0	β^2		0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	3166.908	a +	0	β^2		29	a +	0	β^2		0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	3166.908	a +	0	β^2		29	a +	0	β^2		0.9393	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	3166.908	a +	0	β^2		29	a +	0	β^2		0.9393	a +	0	β^2

Table65:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Normal Distribution for population III

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	309.9455	a +	0	β^2	1	16	a +	0	β^2	2	0.935397	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	270.3297	a +	0.080318	β^2		17.38381	a +	0.080318	β^2		1.193971	a +	0.080318	β^2
		$\hat{\tau}_{g,c=2}$	270.3297	a +	0.080318	β^2		17.38381	a +	0.080318	β^2		1.193971	a +	0.080318	β^2
		$\hat{\tau}_{g,c=3}$	270.3297	a +	0.080318	β^2		17.38381	a +	0.080318	β^2		1.193971	a +	0.080318	β^2
		$\hat{\tau}_{g,c=4}$	270.3297	a +	0.080318	β^2		17.38381	a +	0.080318	β^2		1.193971	a +	0.080318	β^2
0	0.1	$\hat{\tau}_c$	309.9455	a +	0	β^2	1	16	a +	0	β^2	2	0.935397	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	266.0149	a +	0.065452	β^2		16.84624	a +	0.065452	β^2		1.144693	a +	0.065452	β^2
		$\hat{\tau}_{g,c=2}$	269.8331	a +	0.078741	β^2		17.326	a +	0.078741	β^2		1.188758	a +	0.078741	β^2
		$\hat{\tau}_{g,c=3}$	270.2794	a +	0.080159	β^2		17.37799	a +	0.080159	β^2		1.193447	a +	0.080159	β^2
		$\hat{\tau}_{g,c=4}$	270.3247	a +	0.080302	β^2		17.38323	a +	0.080302	β^2		1.193918	a +	0.080302	β^2
0	0.5	$\hat{\tau}_c$	309.9455	a +	0	β^2	1	16	a +	0	β^2	2	0.935397	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	263.614	a +	0.022441	β^2		15.52963	a +	0.022441	β^2		1.002413	a +	0.022441	β^2
		$\hat{\tau}_{g,c=2}$	262.2456	a +	0.046533	β^2		16.20042	a +	0.046533	β^2		1.081737	a +	0.046533	β^2
		$\hat{\tau}_{g,c=3}$	265.1611	a +	0.062033	β^2		16.72554	a +	0.062033	β^2		1.133323	a +	0.062033	β^2
		$\hat{\tau}_{g,c=4}$	267.4639	a +	0.070797	β^2		17.03741	a +	0.070797	β^2		1.162444	a +	0.070797	β^2
0	0.51	$\hat{\tau}_c$	309.9455	a +	0	β^2	1	16	a +	0	β^2	2	0.935397	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	263.8905	a +	0.021649	β^2		15.51342	a +	0.021649	β^2		0.999858	a +	0.021649	β^2
		$\hat{\tau}_{g,c=2}$	262.1098	a +	0.04539	β^2		16.16357	a +	0.04539	β^2		1.077935	a +	0.04539	β^2
		$\hat{\tau}_{g,c=3}$	264.9177	a +	0.061009	β^2		16.68967	a +	0.061009	β^2		1.129916	a +	0.061009	β^2
		$\hat{\tau}_{g,c=4}$	267.2529	a +	0.070047	β^2		17.01041	a +	0.070047	β^2		1.159955	a +	0.070047	β^2

0	0.9	$\hat{\tau}_c$	309.9455	a +	0	β²	1	16	a +	0	β²	2	0.935397	a +	0	β²
		$\hat{\tau}_{g,c=1}$	293.5411	a +	0.00118	β²		15.64636	a +	0.00118	β²		0.937744	a +	0.00118	β²
		$\hat{\tau}_{g,c=2}$	282.6542	a +	0.00393	β²		15.46246	a +	0.00393	β²		0.945173	a +	0.00393	β²
		$\hat{\tau}_{g,c=3}$	275.2801	a +	0.007513	β²		15.38519	a +	0.007513	β²		0.955621	a +	0.007513	β²
		$\hat{\tau}_{g,c=4}$	270.2481	a +	0.011528	β²		15.37709	a +	0.011528	β²		0.967828	a +	0.011528	β²
0	1	$\hat{\tau}_c$	309.9455	a +	0	β²	1	16	a +	0	β²	2	0.935397	a +	0	β²
		$\hat{\tau}_{g,c=1}$	309.9455	a +	0	β²		16	a +	0	β²		0.935397	a +	0	β²
		$\hat{\tau}_{g,c=2}$	309.9455	a +	0	β²		16	a +	0	β²		0.935397	a +	0	β²
		$\hat{\tau}_{g,c=3}$	309.9455	a +	0	β²		16	a +	0	β²		0.935397	a +	0	β²
		$\hat{\tau}_{g,c=4}$	309.9455	a +	0	β²		16	a +	0	β²		0.935397	a +	0	β²

Table66:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical chi square Distribution for population III

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	403234.6	a +	0	β^2	1	16	a +	0	β^2	2	0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	263.1843	a +	0.152949	β^2		23.35668	a +	0.152949	β^2		2.21809	a +	0.152949	β^2
		$\hat{\tau}_{g,c=2}$	263.1843	a +	0.152949	β^2		23.35668	a +	0.152949	β^2		2.21809	a +	0.152949	β^2
		$\hat{\tau}_{g,c=3}$	263.1843	a +	0.152949	β^2		23.35668	a +	0.152949	β^2		2.21809	a +	0.152949	β^2
		$\hat{\tau}_{g,c=4}$	263.1843	a +	0.152949	β^2		23.35668	a +	0.152949	β^2		2.21809	a +	0.152949	β^2
0	0.1	$\hat{\tau}_c$	403234.6	a +	0	β^2	1	16	a +	0	β^2	2	0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	239.4568	a +	0.116728	β^2		20.7772	a +	0.116728	β^2		1.953667	a +	0.116728	β^2
		$\hat{\tau}_{g,c=2}$	260.5395	a +	0.148842	β^2		23.07283	a +	0.148842	β^2		2.189023	a +	0.148842	β^2
		$\hat{\tau}_{g,c=3}$	262.9169	a +	0.152533	β^2		23.32801	a +	0.152533	β^2		2.215155	a +	0.152533	β^2
		$\hat{\tau}_{g,c=4}$	263.1575	a +	0.152907	β^2		23.35381	a +	0.152907	β^2		2.217796	a +	0.152907	β^2
0	0.32	$\hat{\tau}_c$	403234.6	a +	0	β^2	1	16	a +	0	β^2	2	0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	204.6528	a +	0.064235	β^2		16.66747	a +	0.064235	β^2		1.528718	a +	0.064235	β^2
		$\hat{\tau}_{g,c=2}$	238.9565	a +	0.115977	β^2		20.72179	a +	0.115977	β^2		1.947977	a +	0.115977	β^2
		$\hat{\tau}_{g,c=3}$	254.7554	a +	0.139925	β^2		22.44937	a +	0.139925	β^2		2.12516	a +	0.139925	β^2
		$\hat{\tau}_{g,c=4}$	260.4126	a +	0.148645	β^2		23.0592	a +	0.148645	β^2		2.187627	a +	0.148645	β^2
0	0.51	$\hat{\tau}_c$	403234.6	a +	0	β^2	1	16	a +	0	β^2	2	0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	190.5905	a +	0.0366	β^2		14.30043	a +	0.0366	β^2		1.27604	a +	0.0366	β^2
		$\hat{\tau}_{g,c=2}$	212.0788	a +	0.075779	β^2		17.6153	a +	0.075779	β^2		1.627522	a +	0.075779	β^2
		$\hat{\tau}_{g,c=3}$	232.9032	a +	0.106926	β^2		20.04655	a +	0.106926	β^2		1.878588	a +	0.106926	β^2
		$\hat{\tau}_{g,c=4}$	246.5	a +	0.127344	β^2		21.55191	a +	0.127344	β^2		2.033165	a +	0.127344	β^2

0	0.9	$\hat{\tau}_c$	403234.6	a +	0	β^2	1	16	a +	0	β^2	2	0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	279.6471	a +	0.004675	β^2		12.08126	a +	0.004675	β^2		0.954416	a +	0.004675	β^2
		$\hat{\tau}_{g,c=2}$	218.6081	a +	0.010113	β^2		12.19525	a +	0.010113	β^2		1.009845	a +	0.010113	β^2
		$\hat{\tau}_{g,c=3}$	199.2812	a +	0.015843	β^2		12.5611	a +	0.015843	β^2		1.068948	a +	0.015843	β^2
		$\hat{\tau}_{g,c=4}$	191.8762	a +	0.02179	β^2		13.02836	a +	0.02179	β^2		1.129732	a +	0.02179	β^2
0	1	$\hat{\tau}_c$	403234.6	a +	0	β^2	1	16	a +	0	β^2	2	0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	403234.6	a +	0	β^2		16	a +	0	β^2		0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	403234.6	a +	0	β^2		16	a +	0	β^2		0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	403234.6	a +	0	β^2		16	a +	0	β^2		0.910672	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	403234.6	a +	0	β^2		16	a +	0	β^2		0.910672	a +	0	β^2

Table67:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Gamma Distribution for population III

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	408.1397	a +	0	β^2	1	16	a +	0	β^2	2	0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	267.1293	a +	0.225082	β^2		19.99049	a +	0.225082	β^2		1.707981	a +	0.225082	β^2
		$\hat{\tau}_{g,c=2}$	267.1293	a +	0.225082	β^2		19.99049	a +	0.225082	β^2		1.707981	a +	0.225082	β^2
		$\hat{\tau}_{g,c=3}$	267.1293	a +	0.225082	β^2		19.99049	a +	0.225082	β^2		1.707981	a +	0.225082	β^2
		$\hat{\tau}_{g,c=4}$	267.1293	a +	0.225082	β^2		19.99049	a +	0.225082	β^2		1.707981	a +	0.225082	β^2
0	0.1	$\hat{\tau}_c$	408.1397	a +	0	β^2	1	16	a +	0	β^2	2	0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	254.8876	a +	0.174802	β^2		18.37435	a +	0.174802	β^2		1.538015	a +	0.174802	β^2
		$\hat{\tau}_{g,c=2}$	265.6993	a +	0.21948	β^2		19.81166	a +	0.21948	β^2		1.689342	a +	0.21948	β^2
		$\hat{\tau}_{g,c=3}$	266.9841	a +	0.224515	β^2		19.97242	a +	0.224515	β^2		1.7061	a +	0.224515	β^2
		$\hat{\tau}_{g,c=4}$	267.1148	a +	0.225025	β^2		19.98869	a +	0.225025	β^2		1.707793	a +	0.225025	β^2
0	0.32	$\hat{\tau}_c$	408.1397	a +	0	β^2	1	16	a +	0	β^2	2	0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	242.293	a +	0.097498	β^2		15.90531	a +	0.097498	β^2		1.264434	a +	0.097498	β^2
		$\hat{\tau}_{g,c=2}$	254.6474	a +	0.173738	β^2		18.33994	a +	0.173738	β^2		1.53435	a +	0.173738	β^2
		$\hat{\tau}_{g,c=3}$	262.6203	a +	0.207236	β^2		19.41955	a +	0.207236	β^2		1.648347	a +	0.207236	β^2
		$\hat{\tau}_{g,c=4}$	265.6311	a +	0.219212	β^2		19.80308	a +	0.219212	β^2		1.688447	a +	0.219212	β^2
0	0.51	$\hat{\tau}_c$	408.1397	a +	0	β^2	1	16	a +	0	β^2	2	0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	246.8547	a +	0.053877	β^2		14.68803	a +	0.053877	β^2		1.104994	a +	0.053877	β^2
		$\hat{\tau}_{g,c=2}$	243.883	a +	0.115055	β^2		16.45269	a +	0.115055	β^2		1.327845	a +	0.115055	β^2
		$\hat{\tau}_{g,c=3}$	251.8237	a +	0.160816	β^2		17.92204	a +	0.160816	β^2		1.48963	a +	0.160816	β^2
		$\hat{\tau}_{g,c=4}$	258.362	a +	0.189755	β^2		18.85717	a +	0.189755	β^2		1.589201	a +	0.189755	β^2

0	0.9	$\hat{\tau}_c$	408.1397	a +	0	β^2	1	16	a +	0	β^2	2	0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	336.9437	a +	0.003625	β^2		14.87091	a +	0.003625	β^2		0.932355	a +	0.003625	β^2
		$\hat{\tau}_{g,c=2}$	299.0812	a +	0.011033	β^2		14.39677	a +	0.011033	β^2		0.954576	a +	0.011033	β^2
		$\hat{\tau}_{g,c=3}$	276.7375	a +	0.019979	β^2		14.24224	a +	0.019979	β^2		0.984073	a +	0.019979	β^2
		$\hat{\tau}_{g,c=4}$	262.8367	a +	0.029656	β^2		14.26646	a +	0.029656	β^2		1.017636	a +	0.029656	β^2
0	1	$\hat{\tau}_c$	408.1397	a +	0	β^2	1	16	a +	0	β^2	2	0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	408.1397	a +	0	β^2		16	a +	0	β^2		0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	408.1397	a +	0	β^2		16	a +	0	β^2		0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	408.1397	a +	0	β^2		16	a +	0	β^2		0.924323	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	408.1397	a +	0	β^2		16	a +	0	β^2		0.924323	a +	0	β^2

Table68:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Normal Distribution for population IV

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	340.2711	a +	0	β^2	1	16	a +	0	β^2	2	0.863202	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	345.281	a +	0.105493	β^2		20.39472	a +	0.105493	β^2		1.275766	a +	0.105493	β^2
		$\hat{\tau}_{g,c=2}$	345.281	a +	0.105493	β^2		20.39472	a +	0.105493	β^2		1.275766	a +	0.105493	β^2
		$\hat{\tau}_{g,c=3}$	345.281	a +	0.105493	β^2		20.39472	a +	0.105493	β^2		1.275766	a +	0.105493	β^2
		$\hat{\tau}_{g,c=4}$	345.281	a +	0.105493	β^2		20.39472	a +	0.105493	β^2		1.275766	a +	0.105493	β^2
0	0.1	$\hat{\tau}_c$	340.2711	a +	0	β^2	1	16	a +	0	β^2	2	0.863202	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	333.9158	a +	0.085473	β^2		19.44206	a +	0.085473	β^2		1.204443	a +	0.085473	β^2
		$\hat{\tau}_{g,c=2}$	344.0406	a +	0.103351	β^2		20.29305	a +	0.103351	β^2		1.268205	a +	0.103351	β^2
		$\hat{\tau}_{g,c=3}$	345.1559	a +	0.105278	β^2		20.38448	a +	0.105278	β^2		1.275005	a +	0.105278	β^2
		$\hat{\tau}_{g,c=4}$	345.2684	a +	0.105472	β^2		20.39369	a +	0.105472	β^2		1.27569	a +	0.105472	β^2
0	0.775	$\hat{\tau}_c$	340.2711	a +	0	β^2	1	16	a +	0	β^2	2	0.863202	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	314.4131	a +	0.007303	β^2		15.98776	a +	0.007303	β^2		0.90676	a +	0.007303	β^2
		$\hat{\tau}_{g,c=2}$	309.0091	a +	0.019926	β^2		16.42917	a +	0.019926	β^2		0.958809	a +	0.019926	β^2
		$\hat{\tau}_{g,c=3}$	310.4453	a +	0.033057	β^2		16.98626	a +	0.033057	β^2		1.009936	a +	0.033057	β^2
		$\hat{\tau}_{g,c=4}$	314.3506	a +	0.045325	β^2		17.54289	a +	0.045325	β^2		1.056594	a +	0.045325	β^2

0	0.9	$\hat{\tau}_c$	340.2711	a +	0	β²	1	16	a +	0	β²	2	0.863202	a +	0	β²
		$\hat{\tau}_{g,c=1}$	325.2462	a +	0.001664	β²		15.8957	a +	0.001664	β²		0.879025	a +	0.001664	β²
		$\hat{\tau}_{g,c=2}$	316.7422	a +	0.005399	β²		15.94085	a +	0.005399	β²		0.898166	a +	0.005399	β²
		$\hat{\tau}_{g,c=3}$	312.0518	a +	0.010142	β²		16.0718	a +	0.010142	β²		0.919003	a +	0.010142	β²
		$\hat{\tau}_{g,c=4}$	309.738	a +	0.015374	β²		16.25346	a +	0.015374	β²		0.940586	a +	0.015374	β²
0	1	$\hat{\tau}_c$	340.2711	a +	0	β²	1	16	a +	0	β²	2	0.863202	a +	0	β²
		$\hat{\tau}_{g,c=1}$	340.2711	a +	0	β²		16	a +	0	β²		0.863202	a +	0	β²
		$\hat{\tau}_{g,c=2}$	340.2711	a +	0	β²		16	a +	0	β²		0.863202	a +	0	β²
		$\hat{\tau}_{g,c=3}$	340.2711	a +	0	β²		16	a +	0	β²		0.863202	a +	0	β²
		$\hat{\tau}_{g,c=4}$	340.2711	a +	0	β²		16	a +	0	β²		0.863202	a +	0	β²

Table69:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical chi square Distribution for population IV

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	1614.477	a +	0	β^2	1	16	a +	0	β^2	2	0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	356.0437	a +	0.771814	β^2		37.5296	a +	0.771814	β^2		4.656567	a +	0.771814	β^2
		$\hat{\tau}_{g,c=2}$	356.0437	a +	0.771814	β^2		37.5296	a +	0.771814	β^2		4.656567	a +	0.771814	β^2
		$\hat{\tau}_{g,c=3}$	356.0437	a +	0.771814	β^2		37.5296	a +	0.771814	β^2		4.656567	a +	0.771814	β^2
		$\hat{\tau}_{g,c=4}$	356.0437	a +	0.771814	β^2		37.5296	a +	0.771814	β^2		4.656567	a +	0.771814	β^2
0	0.1	$\hat{\tau}_c$	1614.477	a +	0	β^2	1	16	a +	0	β^2	2	0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	294.6133	a +	0.470549	β^2		28.77923	a +	0.470549	β^2		3.450415	a +	0.470549	β^2
		$\hat{\tau}_{g,c=2}$	348.3631	a +	0.732278	β^2		36.44641	a +	0.732278	β^2		4.506554	a +	0.732278	β^2
		$\hat{\tau}_{g,c=3}$	355.2567	a +	0.767739	β^2		37.41869	a +	0.767739	β^2		4.641199	a +	0.767739	β^2
		$\hat{\tau}_{g,c=4}$	355.9648	a +	0.771405	β^2		37.51848	a +	0.771405	β^2		4.655027	a +	0.771405	β^2
0	0.775	$\hat{\tau}_c$	1614.477	a +	0	β^2	1	16	a +	0	β^2	2	0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	292.1775	a +	0.027213	β^2		12.0722	a +	0.027213	β^2		1.069921	a +	0.027213	β^2
		$\hat{\tau}_{g,c=2}$	229.7782	a +	0.06192	β^2		13.39109	a +	0.06192	β^2		1.322983	a +	0.06192	β^2
		$\hat{\tau}_{g,c=3}$	220.6601	a +	0.105742	β^2		15.37721	a +	0.105742	β^2		1.614858	a +	0.105742	β^2
		$\hat{\tau}_{g,c=4}$	226.5605	a +	0.158389	β^2		17.659	a +	0.158389	β^2		1.931919	a +	0.158389	β^2

0	0.9	$\hat{\tau}_c$	1614.477	a +	0	β^2	1	16	a +	0	β^2	2	0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	468.2087	a +	0.01041	β^2		12.27262	a +	0.01041	β^2		0.948087	a +	0.01041	β^2
		$\hat{\tau}_{g,c=2}$	320.3719	a +	0.022027	β^2		11.99157	a +	0.022027	β^2		1.031314	a +	0.022027	β^2
		$\hat{\tau}_{g,c=3}$	266.2546	a +	0.034768	β^2		12.28356	a +	0.034768	β^2		1.12628	a +	0.034768	β^2
		$\hat{\tau}_{g,c=4}$	240.9233	a +	0.048899	β^2		12.82387	a +	0.048899	β^2		1.230122	a +	0.048899	β^2
0	1	$\hat{\tau}_c$	1614.477	a +	0	β^2	1	16	a +	0	β^2	2	0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	1614.477	a +	0	β^2		16	a +	0	β^2		0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	1614.477	a +	0	β^2		16	a +	0	β^2		0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	1614.477	a +	0	β^2		16	a +	0	β^2		0.890109	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	1614.477	a +	0	β^2		16	a +	0	β^2		0.890109	a +	0	β^2

Table70:Expected MSE of linear alternative estimators as compared with that of HHE for the theoretical Gamma Distribution for population IV

G	Rho	Estimator	A			β^2	G	A			β^2	g	A			β^2
0	0	$\hat{\tau}_c$	1072.545	a +	0	β^2	1	16	a +	0	β^2	2	0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	346.5532	a +	2.018657	β^2		42.03505	a +	2.018657	β^2		6.816637	a +	2.018657	β^2
		$\hat{\tau}_{g,c=2}$	346.5532	a +	2.018657	β^2		42.03505	a +	2.018657	β^2		6.816637	a +	2.018657	β^2
		$\hat{\tau}_{g,c=3}$	346.5532	a +	2.018657	β^2		42.03505	a +	2.018657	β^2		6.816637	a +	2.018657	β^2
		$\hat{\tau}_{g,c=4}$	346.5532	a +	2.018657	β^2		42.03505	a +	2.018657	β^2		6.816637	a +	2.018657	β^2
0	0.1	$\hat{\tau}_c$	1072.545	a +	0	β^2	1	16	a +	0	β^2	2	0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	278.0059	a +	1.004935	β^2		28.90207	a +	1.004935	β^2		4.330252	a +	1.004935	β^2
		$\hat{\tau}_{g,c=2}$	337.0724	a +	1.866924	β^2		40.22667	a +	1.866924	β^2		6.468924	a +	1.866924	β^2
		$\hat{\tau}_{g,c=3}$	345.5691	a +	2.002736	β^2		41.84726	a +	2.002736	β^2		6.780459	a +	2.002736	β^2
		$\hat{\tau}_{g,c=4}$	346.4544	a +	2.017057	β^2		42.0162	a +	2.017057	β^2		6.813005	a +	2.017057	β^2
0	0.775	$\hat{\tau}_c$	1072.545	a +	0	β^2	1	16	a +	0	β^2	2	0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	329.7833	a +	0.037274	β^2		11.73501	a +	0.037274	β^2		1.032284	a +	0.037274	β^2
		$\hat{\tau}_{g,c=2}$	248.202	a +	0.091583	β^2		12.45643	a +	0.091583	β^2		1.298998	a +	0.091583	β^2
		$\hat{\tau}_{g,c=3}$	227.0306	a +	0.165763	β^2		14.09383	a +	0.165763	β^2		1.631728	a +	0.165763	β^2
		$\hat{\tau}_{g,c=4}$	224.6873	a +	0.262635	β^2		16.20914	a +	0.262635	β^2		2.021227	a +	0.262635	β^2

0	0.9	$\hat{\tau}_c$	1072.545	a +	0	β^2	1	16	a +	0	β^2	2	0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	506.4554	a +	0.012648	β^2		12.5379	a +	0.012648	β^2		0.914667	a +	0.012648	β^2
		$\hat{\tau}_{g,c=2}$	361.7329	a +	0.029527	β^2		11.80297	a +	0.029527	β^2		0.993887	a +	0.029527	β^2
		$\hat{\tau}_{g,c=3}$	298.4913	a +	0.048743	β^2		11.76907	a +	0.048743	β^2		1.089624	a +	0.048743	β^2
		$\hat{\tau}_{g,c=4}$	264.9976	a +	0.070729	β^2		12.05963	a +	0.070729	β^2		1.198528	a +	0.070729	β^2
0	1	$\hat{\tau}_c$	1072.545	a +	0	β^2	1	16	a +	0	β^2	2	0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=1}$	1072.545	a +	0	β^2		16	a +	0	β^2		0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=2}$	1072.545	a +	0	β^2		16	a +	0	β^2		0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=3}$	1072.545	a +	0	β^2		16	a +	0	β^2		0.866383	a +	0	β^2
		$\hat{\tau}_{g,c=4}$	1072.545	a +	0	β^2		16	a +	0	β^2		0.866383	a +	0	β^2