Journal of Algebra **237,** 487–500 (2001)
doi:10.1006/jabr.2000.8589, available online at http://www.idealibrary.com on IDEAL®

Units of Burnside Rings of Elementary Abelian 2-Groups

Michael A. Alawode

Department of Mathematics, University of Ibadan, Ibadan, Nigeria

Communicated by Walter Feit

Received June 7, 1999

INTRODUCTION

Let G be a finite group. Then the set $S(G)$ of G-isomorphism classes of all finite (left) G-sets forms a semi-ring under addition and multiplication induced, respectively, by the disjoint union and cartesian product. The Grothendieck ring of $S(G)$ is called the Burnside ring of G and is denoted by $\Omega(G)$. Let $\Omega(G)^*$ be the group of units of the Burnside ring of G.

Let G be an elementary Abelian 2-group. In Section 1 of this paper, we study subgroups of the character group

$$\mathrm{Char}(G) = \{\chi\colon G \to \{\pm 1\} \mid \chi(a_1 \cdot a_2) = \chi(a_1)\chi(a_2)\ \forall\ a_1, a_2 \in G\}$$

and prove the following main result:

If

$$\underline{\overline{X}} \subset \mathrm{Char}(G) \qquad \text{and for} \qquad a_1, a_2, \ldots, a_k \in G,$$

put

$$\underline{\overline{X}}(a_1, a_2, \ldots; a_k) := \left\{\chi \in \underline{\overline{X}} \mid \chi(a_1) = \cdots = \chi(a_k) = 1\right\};$$

then the following are equivalent:

(a) $\#\{\chi \in \underline{\overline{X}} \mid {}^{\chi}|_U = 1_{\mathrm{Char}(U)}$ for all subgroups $U \leq G$ with $|U| \leq 2^k\} \equiv 0 \bmod 2$.

(b) $\#\{\chi \in \underline{\overline{X}} \mid \chi(a_1) = \cdots = \chi(a_k) = 1\ \forall\ a_1, \ldots, a_k \in G\} \equiv 0 \bmod 2$.

(c) $\#\underline{\overline{X}}(a_1, a_2, \ldots, a_k) \equiv 0 \bmod 2$ for all $a_1, a_2, \ldots, a_k \in G$ with $l(a_i) \leq 1$ for all $i = 1, 2, \ldots, k$.

0021-8693/01 $35.00

In Section 2, we study $\Omega(G)^*$ as a Burnside ring module. First we identify the group $\rho(\mathrm{Char}(G))$ (formed by the power set of Char(G) under symmetric difference) with $\Omega(G)^*$ as a subgroup of $\{\pm 1\}^{\mathrm{Sub}(G)}$, where Sub($G$) = conjugacy classes of subgroups. This is done through a map

$$\eta\colon \rho(\mathrm{Char}(G)) \to \Omega(G)^* \subseteq \{\pm 1\}^{\mathrm{Sub}(G)}$$

given by $\eta(\underline{\overline{X}}) = M_{\underline{\overline{X}}}$, where $M_{\underline{\overline{X}}}\colon \mathrm{Sub}(G) \to \{\pm 1\}$ is given by

$$M_{\underline{\overline{X}}}(H) = (-1)^{\#\{\chi \in \underline{\overline{X}} \mid \chi(h)=1\ \forall\ h \in H\}}.$$

We then obtain a filtration of $\Omega(G)^*$ for $|G| = 2^n$:

$$\Omega(G)^* = \Omega_{-1}(G)^* \supset \Omega_0(G)^* \supset \cdots \supset \Omega_n(G)^*.$$

1. SUBGROUPS OF CHARACTER GROUP (Char(G)) OF ELEMENTARY ABELIAN 2-GROUPS G

Let

$$N := \{1, 2, 3, \dots\},\ N_0 := \{0\} \cup N, \qquad n \in N_0.$$

Let

$$G := \{\pm 1\}^n = \{(\epsilon_1, \dots, \epsilon_n) \mid \epsilon_i \in \{\pm 1\}\}$$

be the elementary Abelian 2-group of order 2^n. Note that G is a group relative to componentwise multiplication with $1_G = (1, 1, \dots, 1)$ and

$$G \simeq \underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}.$$

Define a function

$$l\colon G \to N_0$$

by

$$l((\epsilon_1, \dots, \epsilon_n)) = \#\{i \in \{1, \dots, n\} \mid \epsilon_i = -1\}.$$

Call l a length function on G.

Let

$$\mathrm{Char}(G) := \{\chi\colon G \to \{\pm 1\} \mid \chi(a_1 \cdot a_2) = \chi(a_1) \cdot \chi(a_2) \quad \text{for all } a_1, a_2 \in G\}.$$

Then Char(G) is a group relative to argumentwise multiplication with identity character

$$1_{\mathrm{Char}(G)}\colon G \to \{\pm 1\}, \qquad a \mapsto +1.$$

The map

$$\mathrm{Char}(G) \to G$$
$$\chi \mapsto (\chi(e_1), \ldots, \chi(e_n))$$

with

$$e_i := (1, \ldots, 1, -1, 1, \ldots, 1)$$
$$\uparrow$$
$$i\text{th position}$$

for all $i = 1, \ldots, n$ is a group isomorphism.

For $\underline{\overline{X}} \subseteq \mathrm{Char}(G)$ and $a_1, \ldots, a_k \in G$ for some $k \in N_0$, put

$$\underline{\overline{X}}(a_1, \ldots, a_k) := \{\chi \in \underline{\overline{X}} \mid \chi(a_1) = \cdots = \chi(a_k) = 1\}.$$

LEMMA 1.1. *For all $a_1, \ldots, a_{k-1}, b_1, b_2 \in G$ and all $\underline{\overline{X}} \subseteq \mathrm{Char}(G)$ one has*

$$\underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_1 b_2) = \underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_1) \vartriangle \underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_2) \vartriangle \underline{\overline{X}}(a_1, \ldots, a_{k-1}),$$

where for arbitrary sets $\underline{\overline{Y}}, \underline{\overline{Z}}$ one puts

$$\underline{\overline{Y}} \vartriangle \underline{\overline{Z}} := (\underline{\overline{Y}} - \underline{\overline{Z}}) \,\dot{\cup}\, (\underline{\overline{Z}} - \underline{\overline{Y}}),$$

noting that

$$(\underline{\overline{Y}} \vartriangle \underline{\overline{Z}}_1) \vartriangle \underline{\overline{Z}}_2 = \underline{\overline{Y}} \vartriangle (\underline{\overline{Z}}_1 \vartriangle \underline{\overline{Z}}_2).$$

Proof. Because

$$\underline{\overline{X}} = \underline{\overline{Y}} \vartriangle \underline{\overline{Z}} \Leftrightarrow \underline{\overline{Y}} = \underline{\overline{Z}} \vartriangle \underline{\overline{X}},$$

it is enough to verify that

$$\underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_1 \cdot b_2) \vartriangle \underline{\overline{X}}(a_1, \ldots, a_{k-1}) = \underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_1) \vartriangle \underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_2).$$

But

$$\text{L.H.S.} = \{\chi \in \underline{\overline{X}}(a_1, \ldots, a_{k-1}) \mid \chi(b_1 \cdot b_2) = -1\}$$

and

$$\text{R.H.S.} = \left\{ \chi \in \underline{\overline{X}}(a_1, \dots, a_{k-1}) | \chi(b_1) = 1 \text{ and } \chi(b_2) = -1 \text{ or } \chi(b_1) = -1 \text{ and } \chi(b_2) = 1 \right\};$$

hence,

$$\text{L.H.S.} = \text{R.H.S.}$$

∎

LEMMA 1.2. *For arbitrary subsets* $\underline{\overline{Y}}_1, \underline{\overline{Y}}_2, \dots, \underline{\overline{Y}}_n \subseteq \mathrm{Char}(G)$ *one has*

$$\#\left(\underline{\overline{Y}} \vartriangle \underline{\overline{Y}}_2 \vartriangle \cdots \vartriangle \underline{\overline{Y}}_n\right) = \sum_{\phi \neq T \subseteq \{1,2,\dots,n\}} (-2)^{(\#T)^{-1}} \cdot \#\left(\textstyle\bigcap_{i \in T} \underline{\overline{Y}}_i\right).$$

Proof. We shall supply a proof of this by induction. First let us check the formula for two sets, say $\underline{\overline{Y}}, \underline{\overline{Y}}_2$ (see Fig. 1). We have

$$\#\underline{\overline{Y}}_1 = \#\left(\underline{\overline{Y}}_1/\underline{\overline{Y}}_2\right) + \#\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_2\right)$$

$$\#\underline{\overline{Y}}_2 = \#\left(\underline{\overline{Y}}_2/\underline{\overline{Y}}_1\right) + \#\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_2\right)$$

$$\#\underline{\overline{Y}}_1 + \#\underline{\overline{Y}}_2 = \#\left(\underline{\overline{Y}}_1/\underline{\overline{Y}}_2\right) + \#\left(\underline{\overline{Y}}_2/\underline{\overline{Y}}_1\right) + 2\#\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_2\right)$$

$$\#\left(\underline{\overline{Y}}_1/\underline{\overline{Y}}_2 \cup \underline{\overline{Y}}_2/\underline{\overline{Y}}_1\right) = \#\underline{\overline{Y}}_1 + \#\underline{\overline{Y}}_2 - 2\#\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_2\right),$$

and so we have

$$\#\left(\underline{\overline{Y}}_1 \vartriangle \underline{\overline{Y}}_2\right) = \#\underline{\overline{Y}}_1 + \#\underline{\overline{Y}}_2 - 2\#\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_2\right);$$

hence, the formula is true for $n = 2$.

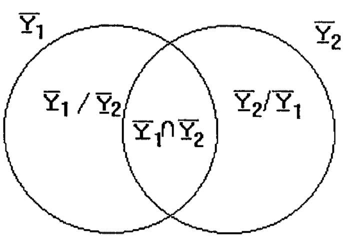

FIGURE 1

Next, let us assume that the formula holds for $n-1$ sets; that is,

$$\#\left(\underline{\overline{Y}}_1 \triangle \underline{\overline{Y}}_2 \triangle \cdots \triangle \underline{\overline{Y}}_{n-1}\right) = \sum_{\phi \neq T \subseteq \{1,2,\ldots,n-1\}} (-2)^{(\#T)-1} \cdot \#\left(\bigcap_{i\in T} \underline{\overline{Y}}_i\right).$$

It follows from our previous result (equivalent to the case $n = 2$) that

$$\begin{aligned}
&\#\left(\underline{\overline{Y}}_1 \triangle \underline{\overline{Y}}_2 \triangle \cdots \triangle \underline{\overline{Y}}_{n-1} \triangle \underline{\overline{Y}}_n\right)\\
&\quad = \#\left[\left(\underline{\overline{Y}}_1 \triangle \underline{\overline{Y}}_2 \triangle \cdots \triangle \underline{\overline{Y}}_{n-1}\right) \triangle \underline{\overline{Y}}_n\right]\\
&\quad = \#\left(\underline{\overline{Y}}_1 \triangle \underline{\overline{Y}}_2 \triangle \cdots \triangle \underline{\overline{Y}}_{n-1}\right) + \#\underline{\overline{Y}}_n - 2\#\left[\left(\underline{\overline{Y}}_1 \triangle \cdots \triangle \underline{\overline{Y}}_{n-1}\right) \cap \underline{\overline{Y}}_n\right]\\
&\quad = \#\left(\underline{\overline{Y}}_1 \triangle \cdots \triangle \underline{\overline{Y}}_{n-1}\right) + \#\underline{\overline{Y}}_n - 2\#\left[\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_n\right) \triangle \cdots \triangle \underline{\overline{Y}}_{n-1} \cap \underline{\overline{Y}}_n)\right],\\
&\quad = \sum_{\phi \neq T \subseteq \{1,2,\ldots,n-1\}} (-2)^{(\#T)-1} \cdot \#\left(\bigcap_{i\in T} \underline{\overline{Y}}_i\right) + \#\underline{\overline{Y}}_n\\
&\qquad - 2\#\left[\left(\underline{\overline{Y}}_1 \cap \underline{\overline{Y}}_n\right) \triangle \cdots \triangle \left(\underline{\overline{Y}}_{n-1} \cap \underline{\overline{Y}}_n\right)\right].
\end{aligned}$$

The first term can be rewritten as

$$\sum_{\phi \neq T \subseteq \{1,2,\ldots,n-1,n\},\ n \notin T} (-2)^{(\#T)-1} \cdot \#\left(\bigcap_{i\in T} \underline{\overline{Y}}_i\right)$$

and the second and third terms together can be rewritten as

$$\sum_{\phi \neq T \subseteq \{1,2,\ldots,n\},\ n \in T} (-2)^{(\#T)-1} \cdot \#\left(\bigcap_{i\in T} \underline{\overline{Y}}_i\right).$$

The above two results yield

$$\sum_{\phi \neq T \subseteq \{1,2,\ldots,n\}} (-2)^{(\#T)-1} \cdot \#\left(\bigcap_{i\in T} \underline{\overline{Y}}_i\right);$$

hence, the formula is true for all n, and therefore by the principle of induction we obtain

$$\#\left(\underline{\overline{Y}}_1 \triangle \cdots \triangle \underline{\overline{Y}}_n\right) = \sum_{\phi \neq T \subseteq \{1,2,\ldots,n\}} (-2)^{(\#T)-1} \cdot \#\left(\bigcap_{i\in T} \underline{\overline{Y}}_i\right).$$

∎

An immediate application of the above formula is as follows. For $n = 2$,

$$\#\left(\underline{\overline{Y}}_1 \triangle \underline{\overline{Y}}_2\right) \equiv \#\underline{\overline{Y}}_1 + \#\underline{\overline{Y}}_2 \mod 2.$$

For $n = 3$,

$$\#\left(\underline{\bar{Y}}_1 \triangle \underline{\bar{Y}}_2 \triangle \underline{\bar{Y}}_3\right) \equiv \#\underline{\bar{Y}}_1 + \#\underline{\bar{Y}}_2 + \#\underline{\bar{Y}}_3 - 2\left[\#\left(\underline{\bar{Y}}_1 \cap \underline{\bar{Y}}_2\right) + \#\left(\underline{\bar{Y}}_1 \cap \underline{\bar{Y}}_3\right) + \#\left(\underline{\bar{Y}}_2 \cap \underline{\bar{Y}}_3\right)\right] \mod 4.$$

For n sets,

$$\#\left(\underline{\bar{Y}}_1 \triangle \cdots \triangle \underline{\bar{Y}}_n\right) \equiv \sum_{i=1}^{n} \#\underline{\bar{Y}}_i - 2 \sum_{1 \le i < j \le n} \#\left(\underline{\bar{Y}}_i \cap \underline{\bar{Y}}_j\right) \mod 4.$$

As a consequence of the above analysis we have

$$\#\left(\underline{\bar{Y}}_1 \triangle \cdots \triangle \underline{\bar{Y}}_n\right) \equiv \sum_{i=1}^{n} \#\underline{\bar{Y}}_i \mod 2.$$

Moreover, since

$$\#\left(\underline{\bar{Y}} \triangle \underline{\bar{Z}}\right) \equiv \#\underline{\bar{Y}} + \#\underline{\bar{Z}} \mod 2$$

for all sets $\underline{\bar{Y}}, \underline{\bar{Z}}$, Lemma 1.1 implies the following corollary:

COROLLARY 1.2. *For all* $\underline{\bar{X}} \subseteq \mathrm{Char}(G)$ *and* $a_1, \ldots, a_{k-1}, b_1, b_2 \in G$ *one has*

$$\#\underline{\bar{X}}(a_1, \ldots, a_{k-1}, b_1 b_2) \equiv \#\underline{\bar{X}}(a_1, \ldots, a_{k-1}, 1_G) + \#\underline{\bar{X}}(a_1, \ldots, a_{k-1}, b_1) + \#\underline{\bar{X}}(a_1, \ldots, a_{k-1}, b_2) \mod 2.$$

THEOREM 1.3. *If* $\underline{\bar{X}} \subseteq \mathrm{Char}(G)$, *then the following are equivalent.*

(a) $\#\{\chi \in \underline{\bar{X}} \mid \chi|_U = 1_{\mathrm{Char}(U)}$, *for all subgroups* $U \le G$ *with* $|U| \le 2^k\} \equiv 0 \bmod 2$.

(b) $\#\{\chi \in \underline{\bar{X}} \mid \chi(a_1) = \cdots = \chi(a_k) = 1$, *for all* $a_1, \ldots, a_k \in G\} \equiv 0 \bmod 2$.

(c) $\#\underline{\bar{X}}(a_1, \ldots, a_k) \equiv 0 \bmod 2$, *for all* $a_1, \ldots, a_k \in G$ *with* $l(a_i) \le 1$ *for all* $i = 1, \ldots, k$.

Proof. Since a subgroup $U \le G$ of G can be generated by k elements $a_1, \ldots, a_k$ from G if and only if $|U| \le 2^k$, (a) $\Leftrightarrow$ (b). It is also clear that (b) $\Rightarrow$ (c).

To show that (c) ⇒ (b) one may proceed by induction relative to

$$l(a_1) + \cdots + l(a_k).$$

If

$$l(a_1) + \cdots + l(a_k) = 0,$$

then

$$l(a_1) = \cdots = l(a_k) = 0$$

and therefore the claim

$$\#\{\chi \in \underline{\overline{X}} \mid \chi(a_1) = \cdots = \chi(a_k) = 1\} \equiv 0 \qquad \mathrm{mod}\, 2$$

follows directly from our assumption.

Now assume that our claim is true whenever

$$l(a_1) + \cdots + l(a_k) \leq n \qquad \text{for some } n \in N$$

and assume that

$$l(a_1) + \cdots + l(a_k) = n + 1 \qquad \text{for some } a_1, \ldots, a_k \in G.$$

If $l(a_i) \leq 1$ for all $i = 1, \ldots, k$, our assumption implies directly that

$$\#\{\chi \in \underline{\overline{X}} \mid \chi(a_1) = \cdots = \chi(a_k) = 1\} \equiv 0 \qquad \mathrm{mod}\, 2.$$

Otherwise $l(a_i) \geq 1$ for some $i \in \{1, \ldots, k\}$, say, $i = k$, so that

$$a_k = b_1 \cdot b_2 \qquad \text{for some } b_1, b_2 \in G, \text{with } l(b_1), l(b_2) < l(a_k)$$

and therefore

$$\sum_{i=1}^{k-1} l(a_i) + l(b_j) \leq n \qquad \text{for } j = 1, 2.$$

Hence

$$\#\underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_1) \equiv 0 \qquad \mathrm{mod}\, 2,$$
$$\#\underline{\overline{X}}(a_1, \ldots, a_{k-1}, b_2) \equiv 0 \qquad \mathrm{mod}\, 2,$$

as well as

$$\#\underline{\overline{X}}(a_1, \ldots, a_{k-1}, 1_G) \equiv 0 \qquad \mathrm{mod}\, 2,$$

by our induction hypothesis, and therefore

$$\begin{aligned}\#\underline{\overline{X}}(a_1,\dots,a_{k-1},a_k) &= \#\underline{\overline{X}}(a_1,\dots,a_{k-1},b_1b_2)\\ &\equiv \#\underline{\overline{X}}(a_1,\dots,a_{k-1},b_1)\\ &\quad + \#\underline{\overline{X}}(a_1,\dots,a_{k-1},b_2)\\ &\quad + \#\underline{\overline{X}}(a_1,\dots,a_{k-1},1_G)\\ &\equiv 0 \qquad \bmod 2 \text{ as claimed.}\end{aligned}$$

2. THE UNIT GROUPS OF BURNSIDE RINGS OF ELEMENTARY ABELIAN 2-GROUPS G AS BURNSIDE RING MODULES

Let G be an elementary Abelian 2-group and let

$$\rho(\mathrm{Char}(G)) := \left\{\underline{\overline{X}} \mid \underline{\overline{X}} \subseteq \mathrm{Char}(G)\right\}$$

be the power set of Char(G).

It can be shown that $\rho(\mathrm{Char}(G))$ is a commutative finite group—more precisely an elementary Abelian 2-group under the symmetric difference $\triangle$ as group multiplication.

For every subset $\underline{\overline{X}} \subseteq \mathrm{Char}(G)$, define

$$M_{\underline{\overline{X}}}: \mathrm{Sub}(G) \to \{\pm 1\}$$

by

$$H \mapsto (-1)^{\#\{\chi \in \underline{\overline{X}} \mid \chi(h)=1, \text{for all } h \in H\}},$$

where Sub(G) denotes the set of subgroups of G.

Let

$$A(G)^* = \left\{M_{\underline{\overline{X}}} \mid \underline{\overline{X}} \subseteq \mathrm{Char}(G)\right\}.$$

Then $A(G)^*$ is a subgroup of the (multiplicative) group of all maps from Sub(G) into $\{\pm 1\}$.

For all $\underline{\overline{X}}, \underline{\overline{Y}} \subseteq \mathrm{Char}(G)$ we have

$$M_{\underline{\overline{X}}} \cdot M_{\underline{\overline{Y}}} = M_{\underline{\overline{X}} \triangle \underline{\overline{Y}}}$$

and

$$M_{\underline{\overline{Y}}} \cdot M_{\underline{\overline{Y}}} = M_{\phi} := \left\{1_{A(G)^*}\right\}.$$

Moreover, $A(G)^*$ can be identified with

$$\Omega(G)^* \subseteq \{\pm 1\}^{\mathrm{Sub}(G)};$$

we henceforth make this identification so that

$$\Omega(G)^* = \left\{M_{\underline{\overline{X}}} \mid \underline{\overline{X}} \subseteq \mathrm{Char}(G)\right\}.$$

THEOREM 2.1. *The map*

$$\rho(\mathrm{Char}(G)) \to \Omega(G)^*$$

defined by

$$\underline{\overline{X}} \to M_{\underline{\overline{X}}}$$

is an isomorphism!

Proof. First, we note that the map

$$\rho(\mathrm{Char}(G)) \to \Omega(G)^*$$

is a well-defined homomorphism.

Second, $\rho(\mathrm{Char}(G))$ and $\Omega(G)^*$ are both of the same order, 2^{2^n}, since $|\mathrm{Char}(G)| = |G| = 2^n$ and since by definition of $\rho(\mathrm{Char}(G))$ as the power set of $\mathrm{Char}(G)$ we have

$$|\rho(\mathrm{Char}(G))| = 2^{2^n},$$

and moreover by standard results of Matsuda [20]

$$|\Omega(G)^*| = 2^{2^n}.$$

We now prove injectivity as follows.

We know that

$$M_{\underline{\overline{X}}} = 1_{\Omega(G)^*} \qquad \text{if and only if } \underline{\overline{X}} = \phi.$$

$M_{\underline{\overline{X}}} = 1_{\Omega(G)^*}$ implies that $M_{\underline{\overline{X}}}(H) = 1$ for all subgroups H of G. $M_{\underline{\overline{X}}}(H) = 1$ if and only if

$$\#\left\{\chi \in \underline{\overline{X}} \mid \chi(h) = 1, \text{ for all } h \in H\right\} \equiv 0 \qquad \mathrm{mod}\, 2$$

and if and only if $\underline{\overline{X}} = \phi$.

We assume that

$$\#\left\{\chi \in \underline{\bar{X}} \mid \chi(h) = 1, \text{for all } h \in H\right\} \equiv 0 \qquad \bmod 2$$

to show first that the trivial character is not in $\underline{\bar{X}}$!

Let χ_1 be the trivial character. We choose for this case the subgroup

$$H := G.$$

By definition of a trivial character we have for an arbitrary character χ that $\chi(h) = 1$ for all $h \in G$ if and only if $\chi = \chi_1$. Hence,

$$\left\{\chi \in \underline{\bar{X}} \mid \chi(h) = 1, \text{for all } h \in G\right\} = \underline{\bar{X}} \cap \{\chi_1\}$$

and therefore

$$\#\left\{\chi \in \underline{\bar{X}} \mid \chi(h) = 1, \text{for all } h \in G\right\}$$
$$= \#\left(\underline{\bar{X}} \cap \{\chi_1\}\right) = \begin{cases} 1, & \text{if } \chi_1 \in \underline{\bar{X}} \\ 0, & \text{if } \chi_1 \notin \underline{\bar{X}}. \end{cases}$$

It follows that

$$\chi_1 \notin \underline{\bar{X}},$$

and so the trivial character is not in $\underline{\bar{X}}$.

Finally, we must show that no non-trivial character is in such an $\underline{\bar{X}}$!

Let χ be a non-trivial character. For such a non-trivial character χ consider

$$H := \{g \in G \mid \chi(g) = 1\},$$

a subgroup of G. For any other non-trivial character χ', $\chi'(g) = 1$ for all $g \in H$ if and only if $\chi = \chi'$. Hence,

$$\left\{\chi' \in \underline{\bar{X}} \mid \chi'(g) = 1, \text{for all } g \in H\right\} = \underline{\bar{X}} \cap \{\chi\}$$

and therefore

$$\#\left\{\chi' \in \underline{\bar{X}} \mid \chi'(g) = 1, \text{for all } g \in H\right\}$$
$$= \#\left(\underline{\bar{X}} \cap \{\chi\}\right) = \begin{cases} 1, & \text{for } \chi \in \underline{\bar{X}} \\ 0, & \text{if } \chi \notin \underline{\bar{X}} \end{cases}$$

and so we have

$$\chi \notin \underline{\overline{X}}.$$

Hence, no non-trivial character is in such an $\underline{\overline{X}}$; therefore, $\underline{\overline{X}} = \phi$, as claimed. Surjectivity follows from all the above considerations. Thus,

$$\rho(\mathrm{Char}(G)) \to \Omega(G)^*$$

is an isomorphism.

Put

$$\Omega_k(G)^* := \{M_{\underline{\overline{X}}} \in \Omega(G)^* | M_{\underline{\overline{X}}}(H) = 1, \text{ for all } H \leq G \text{ with } |H| \leq 2^k\},$$

so that

$$\begin{aligned} \Omega(G)^* = \Omega_{-1}(G)^* \supset \Omega_0(G)^* \supset \Omega_1(G)^* \supset \cdots \supset \Omega_n(G)^* \\ := \{1_{\Omega(G)^*}\} = \{M_\phi\}. \end{aligned}$$

LEMMA 2.2. *$\Omega_k(G)^* = \{M_{\underline{\overline{X}}} \in \Omega_{k-1}(G)^* |$ for all subsets T of $\{1, 2, \ldots, n\}$ of cardinality k; the number of $\chi \in \overline{X}$ with $\chi(e_i) = 1$ for all $i \in T$ is even$\}$.*

LEMMA 2.3.

$$\Omega_k(G)^* = \ker\left(\prod_{T \in \binom{\{1,2,\ldots,n\}}{k}} \lambda_T \colon \Omega_{k-1}(G)^* \to \{\pm 1\}^{\binom{\{1,2,\ldots,n\}}{k}} \right),$$

where

$$\lambda_T \colon \Omega_{k-1}(G)^* \to \{\pm 1\}$$

is the homomorphism which maps every

$$M_{\underline{\overline{X}}} \in \Omega_{k-1}(G)^* \qquad \text{onto } M_{\underline{\overline{X}}}(\langle e_i \mid i \in T \rangle).$$

Proof. Given that $T \subseteq \{1, 2, \ldots, n\}$ with $\#T = k$, consider for each such T the map

$$\begin{aligned} \lambda_T \colon \Omega(G)^* \to \{\pm 1\} \\ : M_{\underline{\overline{X}}} \mapsto (-1)^{\#\{\chi \in \underline{\overline{X}} \mid \chi(e_i) = 1 \text{ for all } i \in T\}}. \end{aligned}$$

We contend that λ_T is a homomorphism!

For $M_{\underline{\bar{X}}}, M_{\underline{\bar{X}}'} \in \Omega(G)^*$, we obtain

$$\lambda_T(M_{\underline{\bar{X}}}) := M_{\underline{\bar{X}}}(\langle e_i \mid i \in T\rangle) := (-1)^{\#\{\chi \in \underline{\bar{X}} \mid \chi(e_i) = 1 \text{ for all } i \in T\}},$$

$$\lambda_T(M_{\underline{\bar{X}}'}) := M_{\underline{\bar{X}}}(\langle e_i \mid i \in T\rangle) := (-1)^{\#\{\chi \in \underline{\bar{X}}' \mid \chi(e_i) = 1 \text{ for all } i \in T\}},$$

$$\lambda_T(M_{\underline{\bar{X}}}) \cdot \lambda_T(M_{\underline{\bar{X}}'}) = (-1)^{\#\{\chi \in \underline{\bar{X}} \mid \chi(e_i) = 1\ \forall i \in T\} + \#\{\chi \in \underline{\bar{X}}' \mid \chi(e_i) = 1\ \forall i \in T\}}$$
$$= (-1)^{\#\{\chi \in \underline{\bar{X}} \vartriangle \underline{\bar{X}}' \mid \chi(e_i) = 1\ \forall i \in T\}},$$

since

$$\#\left(\underline{\bar{X}} \vartriangle \underline{\bar{X}}'\right) \equiv \#\underline{\bar{X}} + \#\underline{\bar{X}}' \quad \mod 2.$$

Hence, by definition, we have

$$\begin{aligned}(-1)&^{\#\{\chi \in \underline{\bar{X}} \vartriangle \underline{\bar{X}}' \mid \chi(e_i) = 1\ \forall i \in T\}} \\ &= M_{\underline{\bar{X}} \vartriangle \underline{\bar{X}}'}(\langle e_i \mid i \in T\rangle) \\ &= M_{\underline{\bar{X}}} \cdot M_{\underline{\bar{X}}'}(\langle e_i \mid i \in T\rangle) \\ &= \lambda_T\left(M_{\underline{\bar{X}}} \cdot M_{\underline{\bar{X}}}\right)\end{aligned}$$

and therefore

$$\lambda_T\left(M_{\underline{\bar{X}}'} \cdot M_{\underline{\bar{X}}}\right) = \lambda_T(M_{\underline{\bar{X}}}) \cdot \lambda_T(M_{\underline{\bar{X}}'})$$

as claimed. Now, since the number of sets of $T \subseteq \{1, 2, \dots, n\}$ with $\#T = k$ is $\binom{n}{k}$, we shall have $\binom{n}{k}$ homomorphisms of such λ_T. Thus

$$\prod_{T \in \binom{\{1,2,\dots,n\}}{k}} \lambda_T \colon \Omega_{k-1}(G)^* \to \{\pm 1\}^{\binom{\{1,2,\dots,n\}}{k}}$$

is also a homomorphism. Therefore by 2.1 and the construction of $\Omega_k(G)^*$ above, we conclude that

$$\Omega_k(G)^* = \ker\left(\prod_{T \in \binom{\{1,2,\dots,n\}}{k}} \lambda_T \colon \Omega_{k-1}(G)^* \to \{\pm 1\}^{\binom{\{1,2,\dots,n\}}{k}}\right).$$

∎

Theorem 2.4.

$$(\Omega_{k-1}(G)^* \colon \Omega_k(G)^*) = 2^{\binom{n}{k}}.$$

APPENDIX: NOMENCLATURE

Throughout this paper we use the following notations:

G is an elementary Abelian 2-group.

$\#X$ or $|X|$ is the cardinal number of a set X.

$1_{\Omega(G)}$ is the unit element [point] of $\Omega(G)$.

R^* is the unit group of a ring R.

$\mathbb{Z}$ is the ring of rational integers.

$\mathbb{Z}_2 := \{\pm 1\}$ is a set having $+1$ and -1 as its elements.

$e_i = (1, \dots, 1, -1, 1, \dots, 1)$ is an element of G, where the ith entry is -1.

$\binom{\{1,2,\dots,n\}}{k}$ is the set of all subsets of order k of the set $A = \{1, 2, \dots, n\}$ where k, n are fixed positive integers, $k \le n$.

Sub(G) is the subgroup lattice of G.

$l: G \to N_0$ is a length function on G_0, where $N_0 := \{0\} \cup N$ is the set of natural numbers N in disjoint union with the singleton set $\{0\}$, having 0 as its only element.

$\triangle$ is the symmetric difference.

ACKNOWLEDGMENTS

I am sincerely grateful to my Ph.D. thesis supervisors Prof. Dr. Andreas Dress and Professor Aderemi O. Kuku for their guidance, patience, and generosity.

REFERENCES

1. M. A. Alawode, The Group of Units of Burnside Rings of Various Finite Groups, Ph.D. thesis, University of Ibadan, Ibadan, Nigeria, 1999.
2. R. Araki, Equivalent stable homotopy theory and idempotents of Burnside rings, *Publ. Res. Inst. Math. Sci.* **18** (1982), 1193–1212.
3. H. Bender, On groups with abelian Sylow 2-subgroups, *Math. Z.* **117** (1970), 164–176.
4. C. W. Curtis and I. Reiner, "Methods of Representation Theory," Vols. 1 and 2, Wiley–Interscience, New York, 1981.
5. T. Dieck, "Transformation Groups and Representation Theory," Lecture Notes in Mathematics, Vol. 766, Springer-Verlag, Berlin/New York, 1979.
6. A. Dress, A characterization of solvable groups, *Math. Z.* **110** (1969), 213–217.
7. A. Dress, Operations in representation rings, *in* "Pro Symposia in Pure Mathematics," pp. 39–45, 1971.
8. A. Dress, Contributions to the theory of induced representations, *in* "Algebraic *K*-Theory II, Proceedings of the Battle Institute Conference, 1972," Lecture Notes in Mathematics, Vol. 342, pp. 183–240, Springer-Verlag, Berlin/New York, 1973.

9. A. Dress, Notes on the theory of representations of finite groups, *Bielefeld Notes* (1971).
10. A. Dress and M. Kuchler, Zur Darstellungstheorie endlicher Gruppen I, *Bielefeld Notes* (1970).
11. W. Feit and J. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 775–1029.
12. D. Gluck, Idempotent formula for the Burnside algebra with applications to the P-subgroup simplicial complex, *Illinois J. Math.* **25** (1981), 63–67.
13. D. Gorenstein, "Finite Groups," Harper & Row, New York, 1968.
14. J. A. Green, Axiomatic representation theory for finite groups, *J. Pure Appl. Algebra* **1** (1971), 41–77.
15. W. H. Greub, "Multilinear Algebra," Springer-Verlag, Berlin/Heidelberg/New York, 1967.
16. W. H. Gustafson, Burnside rings which are Gorenstein, *Comm. Algebra* **5** (1977), 1–16.
17. R. Keown, "An Introduction to Group Representation Theory," 1975.
18. A. O. Kuku, Axiomatic theory of induced representation of finite groups, *in* "Group Representation and Its Applications: Les Cours du C.I.M.P.A." (A. O. Kuku, Ed.), 1985.
19. I. Li, Burnside algebra of a finite inverse semigroup, *Zap. Nauchn. Steklov. Inst.* **46** (1974), 41–52; *J. Soviet Math.* **9** (1978), 322–331.
20. T. Matsuda, On the unit groups of Burnside rings, *Japan. J. Math.* (*N.S.*) **8** (1982), 71–93.
21. T. Matsuda, A note on the unit groups of the Burnside rings as Burnside ring modules, *J. Fac. Sci. Shinshu Univ.* **21**(1) (1986).
22. T. Matsuda and T. Miyata, On the unit groups of the Burnside rings of finite groups, *J. Math. Soc. Japan* **35** (1983), 345–354.
23. H. Sasaki, Green correspondence and transfer theorems of Wielandt type for G-functors, *J. Algebra* **79** (1982), 98–120.
24. J. P. Serre, "Linear Representation of Finite Groups," Graduate Texts in Mathematics, Vol. 42, Springer-Verlag, Berlin/Heidelberg/New York.
25. J. H. Walter, Finite groups with abelian Sylow 2-subgroups, *Ann. of Math.* **89** (1969), 405–514.
26. T. Yoshida, Character-theoretic transfer, *J. Algebra* **52** (1978), 1–38.
27. T. Yoshida, On G-functors. I. Transfer theorems for cohomological G-functors, *Hokkaido Math. J.* **9** (1980), 222–257.
28. T. Yoshida, Idempotents of Burnside rings and Dress induction theorem, *J. Algebra* **80** (1983), 90–105.
29. T. Yoshida, Idempotents and transfer theorems of Burnside rings, character rings and span rings, *in* "Algebraic and Topological Theories," pp. 589–615, Kinokuniya, Tokyo, 1985.
30. T. Yoshida, On the unit groups for Burnside rings, *J. Math. Soc. Japan* **42**(1) (1990).