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In honour of Prof. Ekhaguere at 70

A new equivalence relation for the classification of fuzzy subgroups of symmetric group S_4

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Abstract. In this paper a new equivalence relation for classifying the fuzzy subgroups of finite groups is studied.Without any equivalence relation on fuzzy subgroups of group G, the number of fuzzy subgroups is infinite,even for the trivial group.The number of distinct fuzzy subgroups with respect to the new equivalence relation is obtained for S_4.

Keywords: fuzzy subgroups, chains of subgroups, fuzzy equivalence, symmetric group.

1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965(see[13]). The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 (see[15]). Since the first paper by Rosenfeld, researchers have sought to characterize the fuzzy subgroups of various groups. One of the most important problem of fuzzy theory is to classify the fuzzy subgroups of a finite groups. This topic has enjoyed a rapid development in the last few years .

Sulaiman and Abd Ghafur [7] have counted the number of fuzzy subgroups of symmetric group S_2, S_3 and alternating group A_4 .Sulaiman[6] have constructed the fuzzy subgroups of symmetric group S_4 while Tarnauceanu [12] have also computed the number of fuzzy subgroups of symmetric group S_4 by the inclusion -Exclusion Principle. These groups are probably the most important in group theory, because any finite group can be embedded in such a group. They also have remarkable applications in graph theory, in enumerative combinatorics, as well as in many branches of informatics.

2. Preliminaries

Fuzzy set theory was guided by the assumption that classical sets were not natural, appropriate or useful notions in describing the real life problems, because every object encountered in this real physical world carries some degree of fuzziness. Further the concept of grade of membership is not a probabilistic concept.

It is from the generalisation of the concept of a crisp set that the notion of a fuzzy set is derived.Unlike in classical set theory where membership of an element of a set is viewed in binary terms of a bivalent nature(is a member of or is not a member of),this generalisation of classical sets to fuzzy sets allows for elements of a set to partially belong to that set.

DEFINITION 2.1 *A fuzzy subset of a set X is a function $\mu\colon X \to l = [0, 1]$.*

DEFINITION 2.2 *Let G be a group with a multiplicative binary operation and identity e, and let $\mu\colon G \to [0, 1]$ be a fuzzy subset of G. Then μ is said to be a **Fuzzy Subgroup** of G if*

(1) $\mu(xy) \geq min\{\mu(x), \mu(y)\}$,

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(2) $\mu(x^{-1}) \geq \mu(x)$ *for all* $x, y \in G$

The following elementary facts about fuzzy subgroups follow easily from the axioms: $\mu(x) = \mu(x^{-1})$ and $\mu(x) \leq \mu(e)$, for all $x \in G$. Also, μ satisfies conditions (1) and (2) of Definition if and only if $\min\{\mu(x), \mu(y)\} \leq \mu(xy^{-1})$, for all x, y ∈ G.

The set $\{\mu(x)|x \in G\}$ is called the image of μ and is denoted by $\mu(G)$, For each $\alpha \in \mu(G)$, the set $\mu_\alpha := \{x \in G|\mu(x) \geq \alpha\}$ is called a level subset of μ .It follows that μ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G .These subsets allow us to characterize the fuzzy subgroups of G (see [3])

Suppose G is a finite group; then the number of subgroups of G is finite where as the number of level subgroups of a fuzzy subgroup A appears to be infinite. But, since every level subgroup is indeed a subgroup of G, not all these level subgroups are distinct. In this paper, we count the classified distinct fuzzy subgroups of S_4 .

THEOREM 2.3 *[14] Any subgroup H of a group G can be realised as a level subgroup of some fuzzy subgroup of G*

3. Fuzzy Equivalence Relations

Without any equivalence relation on fuzzy subgroups of group$G$, the number of fuzzy subgroups is infinite, even for the trivial group {e}. So we define equivalence relation on the set of all fuzzy subgroups of a given group. In this paper , we use the definition of Murali and Makamba[3,4], Volf [14] and Tarnaceanu and Bentea[10]. They say that μ is equivalent to ν, written as $\mu \sim \nu$, if we have

$$\mu(x) > \mu(y) \Leftrightarrow \nu(x) > \nu(y), \text{ for all } x, y \in G$$

and

$$\mu(x) = 0 \Leftrightarrow \nu(x) = 0, \text{ for all } x \in G.$$

Note that the condition $\mu(x) = 0$ holds if and only if $\nu(x) = 0$ simply says that the supports of μ and ν are equal and two fuzzy subgroups μ, ν of G are said to be distinct if $\mu \nsim \nu$

Let G be a finite group. Then it is well-defined the following action of $Aut(G)$ on FL(G)

$$\rho : FL(G) \times Aut(G) \to FL(G),$$

$$\rho(\mu, f) = \mu \circ f, \text{ for all } (\mu, f) \in FL(G) \times Aut(G)$$

Let us denote by $\approx_\rho$ the equivalence relation on FL(G) induced by ρ, namely

$$\mu \approx_\rho \nu \text{ if and only if there exists } f \in Aut(G) \text{ such that } \nu = \mu \circ f$$

In this paper,it is called a new equivalence relation, Tarnaceanu[11]. There are other different versions of fuzzy equivalence relations in the literature.

The problem of classifying the fuzzy subgroup of finite group G by using a new equivalence relation $\approx$ on the lattice of all fuzzy subgroups of G, its definition has a consistent group theoretical foundation , by involving the knowledge of the automorphism group associated to G. The approach is motivated by the realization that in a theoretical study of fuzzy groups, fuzzy subgroups are distinguished by their level subgroups and not by their images in $[0, 1]$. Consequently, the study of some equivalence relations between the chains of level subgroups of fuzzy groups is very important.

4. Enumerative Technique For The Number of Fuzzy Subgroups

There are various enumeration techniques that are used in the counting of distinct fuzzy subgroups of a finite group. These counting techniques are derived from the interpretation of the definition of fuzzy equivalence relations used.

The equivalence classes are called the orbits of the action,the orbit of a chain $C \in \overline{C}$ is $\{f(C) \mid f \in Aut(G)\}$, while the set of all chains in $\overline{C}$ that are fixed by an automorphism f of G is

$$Fix_{\overline{C}}(f) = \{C \in \overline{C} | f(C) = C\}$$

.

Now, the number N is obtained by applying the Burnsides lemma:

$$N = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} \mid Fix_{\overline{C}}(f) \mid \quad (1)$$

The Burnside's lemma plays an important role in the explicit formula to compute the number of distinct fuzzy subgroups(N) of a finite group G with respect to a certain equivalence relation on the lattice of fuzzy subgroups, induced by an action of the automorphism group $Aut(G)$ associated to G (see[11]).

5. Main Results

Let $C \in Fix_{\overline{C}}(f_\sigma)$, where $C : H_1 \subset H_2 \subset \cdots \subset H_m = S_n$. Then $f_\sigma(C) = C$, that is $f_\sigma(H_i) = H_i$, for all $i = \overline{1, m}$.Then every automorphism of S_n is of the form f_σ with $\sigma \in S_n$ .In fact, for $n \neq 2, 6$ the symmetric group is a complete group. It is well-known that S_4 has 24 elements, more precisely
$S_4 = \{(), (12), (23), (34), (13), (14), (24), (12)(34), (13)(24), (14)(23),$
$(234), (243), (123), (124), (132), (134), (142), (143),$
$(1234), (1243), (1342), (1324), (1432), (1423)\}$
$f_\tau : S_4 \to S_4$
$f_\tau(\sigma) = \tau^{-1}\sigma\tau$

Then every automorphism of S_n is of the form f_σ with $\sigma \in S_n$
$Aut(S_4) = \{f_1, f_2, f_3, f_4, f_5, f_6, f_6, f_7, f_8, f_9, f_{10}, f_{11}, f_{12}, f_{13}, f_{14}, f_{15}, f_{16}, f_{17}, f_{18}, f_{19}, f_{20}, f_{21}, f_{22}, f_{23}, f_{24}\}$

$|Fix_{\overline{C}}(f_{(e)})| = 232$

$|Fix_{\overline{C}}(f_{(12)(34)}| = |Fix_{\overline{C}}(f_{(13)(24))}| = |Fix_{\overline{C}}(f_{(14)(23))}| = 144$

$|Fix_{\overline{C}}(f_{(34)})| = |Fix_{\overline{C}}(f_{(23)})| = |Fix_{\overline{C}}(f_{(24)})| = |Fix_{\overline{C}}(f_{(12)})|$

$|Fix_{\overline{C}}(f_{(13)})| = |Fix_{\overline{C}}(f_{(14)})| = 76$

$|Fix_{\overline{C}}(f_{(234)})| = |Fix_{\overline{C}}(f_{(243)}| = |Fix_{\overline{C}}(f_{(123)})| =$

$|Fix_{\overline{C}}(f_{(124)})| = |Fix_{\overline{C}}(f_{(132)})| = |Fix_{\overline{C}}(f_{(134)})|$

$|Fix_{\overline{C}}(f_{(142)}| = |Fix_{\overline{C}}(f_{(143)}| = 16$

$|Fix_{\overline{C}}(f_{(1234)})| = |Fix_{\overline{C}}(f_{(1243)})| = |Fix_{\overline{C}}(f_{(1342)})|$

$|Fix_{\overline{C}}(f_{(1324)})| = |Fix_{\overline{C}}(f_{(1432)})| = |Fix_{\overline{C}}(f_(1423)) = 40$

$$N = \frac{1}{|Aut(G)|} \sum_{f \in Aut(G)} |\ Fix_{\overline{C}}(f)\ | \tag{2}$$

$$N = \frac{1}{24}(232 + 6(76) + 6(40) + 3(144) + 8(16)) = 62 \tag{3}$$

THEOREM 5.1 *The number N of all distinct fuzzy subgroups with respect to $\approx$ of the symmetric group S_4 is 62*

The set of chains of subgroups of S_4 fixed by the automorphism f can be represented by the cycle structure of S_4. It is very clear that the inner automorphisms of S_4 preserve each conjugacy class in S_4.

PROPOSITION 5.2 *Two elements of S_n are conjugate if and only if they have the same cycle structure.*

It is well-known from classical group theory,an automorphism of a group G permutes the conjugacy classes in G, and the inner automorphisms preserve each conjugacy class.

THEOREM 5.3 *Let α,β of S_n be conjugate,then the set of number of chains of subgroups of S_n for $n \neq 2, 6$ fixed by the automorphism f is equal.*

Proof. It is well- known that two permutations $g_1, g_2 \in S_n$ are conjugate (that is, $g_2 = h^{-1}g_1h$ for some $h \in S_n$) if and only if they have the same cycle structure. An inner automorphism of G is an automorphism of G. The inner automorphisms comprise a normal subgroup of Aut(G), denoted by Inn(G); it is isomorphic to G/Z(G), where Z(G) is the centre of G. An automorphism of S_n for $n \neq 2, 6$ permutes the conjugacy classes in S_n, and the inner automorphisms preserve each conjugacy class. It follows that the set of chains of subgroups of S_4 fixed by the automorphism f are represented by the cycle structure of S_4. ■

6. Conclusion

This study has shown that even with the same groups as shown by the case of the symmetric group S_4, different results for numbers of distinct fuzzy subgroups are obtained depending on the definition of equivalence relations used.

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