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COMBINATORICS OF COUNTING DISTINCT FUZZY SUBGROUPS OF CERTAIN DIHEDRAL GROUP

(Kombinatorik Membilang Subkumpulan Kabur Berbeza bagi Kumpulan Dwihedron Tertentu)

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ABSTRACT

This paper is devoted to counting distinct fuzzy subgroups (DFS) of finite dihedral group D_{2n}, where n is a product of finite number of distinct primes, with respect to the equivalence relation $\approx$. This counting has connections with familiar integer sequence called ordered Bell numbers. Furthermore, a recurrence relation and generating function was derived for counting DFS of D_{2n}.

Keywords: dihedral group; equivalence relation; fuzzy subgroup; Bell Number; generating function

ABSTRAK

Dalam makalah ini dibincangkan tentang pembilangan subkumpulan kabur yang berbeza bagi kumpulan dwihedron D_{2n}, yang n adalah hasil darab terhingga nombor-nombor perdana yang berbeza, bagi hubungan kesetaraan $\approx$. Pembilangan ini berkaitan dengan jujukan integer yang dikenali sebagai nombor Bell bertertib. Tambahan lagi, suatu hubungan jadi semula dan fungsi penjanaan diperoleh untuk membilang subkumpulan kabur berbeza bagi D_{2n}.

Kata kunci: kumpulan dwihedron; hubungan kesetaraan; subkumpulan kabur; nombor Bell; fungsi penjanaan

1. Introduction

Counting distinct fuzzy subgroups (DFS) of finite dihedral groups is a fundamental combinatorics problem. Research had focused on counting DFS of finite dihedral groups with respect to the equivalence relation $\approx$, see Tarnauceanu (2016). However, literature on DFS of dihedral groups D_{2n}, (where n is a product of finite number of distinct primes) is scarce. This research, was therefore designed to establish a recurrence relation and a generating function for counting DFS of D_{2n} with respect to the equivalence relation $\approx$.

The equivalence relation $\approx$ used in our counting is preferred to other equivalence relation as its yields fewer numbers of DFS of dihedral groups D_{2n} and its definition involves more group theoretical properties compared to other equivalence relation known in literature. For classification of fuzzy subgroups using other equivalence relations, see Tarnauceanu (2012), Murali and Makamba (2001; 2003). It is clear from Tarnauceanu (2016), that the formula obtained for counting the number of DFS of finite groups with respect to $\approx$ involves many concepts namely; order of automorphism group, automorphism group structure, group actions, subgroup lattices and Burnside's Lemma. This technique is very numerous, therefore, a bijective correspondence was established between DFS of D_{2n} and the number of chains subgroups which ends in D_{2n}, fixed by a certain element of automorphism group of dihedral group $Aut(D_{2n})$. This number of chains of subgroups which ends in D_{2n}, and fixed by a certain element of $Aut(D_{2n})$ forms patterns that has relationship with a familiar integer sequence called ordered Bell number. Finally, we derive a recurrence relation and generating function for counting DFS of D_{2n} which generalizes the result obtained by Olayiwola and Isyaku (2018).

The paper is divided into five sections. Section one gives the literature and introduces the problem statement as above. In section two, we give some preliminary definitions and in section three we give an overview of the new equivalence relation $\approx$ as introduced in Tarnauceanu (2016). In section four we derive a recurrence relation and generating function for counting DFS of D_{2n}. Finally, the last section gives the conclusion and a summary of our results.

2. Preliminaries

Given an arbitrary non empty set X, a fuzzy set (on X) is a function from X to the unit interval $I : [0,1]$. That is

$$\varphi : X \rightarrow I$$

Let G be a group and F(G) be collection of all fuzzy subsets of G. An element φ of F(G) is said to be a fuzzy subgroup of G if it satisfies the following two conditions:

- $\varphi(xy) \geq \min\{\varphi(x),\varphi(y)\}, \forall x,y \in G.$
- $\varphi(x^{-1}) \geq \varphi(x), \forall x \in G$. In this situation we have $\varphi(x^{-1}) = \varphi(x)$, for any $x \in G$, and $\varphi(e) = \max\{\varphi(G)\} = \sup\{\varphi(G)\}$.

The set $FL(G)$ which consist of all fuzzy subgroups of G forms a lattice with respect to the usual ordering of fuzzy set inclusion called fuzzy subgroup lattice, see Tarnauceanu (2012). For any $\alpha \in [0,1]$ the level subset is defined by

$$\varphi_\alpha = \{x \in G \mid \varphi(x) \geq \alpha\}.$$

Thus, a fuzzy subset φ is a fuzzy subgroup of G if and only if its level subsets are subgroups of G.

Let Ω be an arbitrary non empty set and G be a group. An action or operator ρ of G on Ω is a map $\Omega \times G \rightarrow \Omega$ satisfying the following axioms.

- $\rho(\omega,g_1,g_2) = \rho(\rho(\omega,g_1),g_2), \forall g_1,g_2 \in G$ and $\omega \in \Omega$;
- $\rho(\omega,e) = \omega, \forall\omega \in \Omega$.

Group actions generalize group multiplication. If G acts or operates on Ω so does any subgroup of G. Ω is called a G-set. Notice that it is not necessary for Ω to be related to G in any way. However, group actions are more interesting if the set Ω is somehow related to the group G.

It is well known from literature that, every group action ρ of G induces an equivalence relation R_ρ on Ω defined by $\omega R_\rho v$ if and only if $\exists\, g \in G$ such that $v = \rho(\omega,g)$. The quotient set with respect to action ρ, is called the orbits of Ω. Let

$$Fix_\Omega(g) = \{\omega \in \Omega \mid \rho(\omega,g) = \omega\}$$

be the set of elements of Ω that are fixed by g. If both G and Ω are finite, then the number of distinct orbits of Ω relative to ρ is given by;

$$N = \frac{1}{|G|}\sum_{g\in G}|Fix_{\Omega}(g)|$$

known as Burnside's Lemma. This results has been applied by Tarnauceanu (2016) for counting the number of distinct fuzzy subgroups of some finite groups G.

3. Overview of the New Equivalence Relation on $FL(G)$

Let G be a finite group, then the action of $Aut(G)$ on $FL(G)$ is given by

$$\rho : FL(G) \times Aut(G) \to FL(G),$$

where $\rho(\mu, f) = \mu f$, is well defined for every $\mu \in FL(G)$ and $f \in Aut(G)$.

This action induces an equivalence relation on $FL(G)$. Since every fuzzy subgroup of G determines a chain of subgroups of G which ends in G, then this action can be seen in terms of chains of subgroups of G. Let $\overline{C}$ be the set of chains of subgroups of G terminated at G, then the previous action of $Aut(G)$ on $FL(G)$ can be seen as action of $Aut(G)$ on $\overline{C}$ and the previous equivalence relation is seen as equivalence relation induced by this action. An equivalence relation is then defined on $\overline{C}$ in the following manner: For two chains,

$$C_1 : H_1 \subset H_2 \subset \cdots \subset H_m = G \text{ and } C_2 : K_1 \subset K_2 \subset \cdots \subset K_n = G$$

of $\overline{C}$, we set

$$C_1 \approx C_2 \Leftrightarrow m = n \text{ and } \exists\, f \in Aut(G) \text{ such that } f(H_i) = K_i \ \forall i = 1,2,\cdots,n.$$

The orbit of a chain $C \in \overline{C}$ is now given by $\{ f(C) \mid f \in Aut(G)\}$, and the set of all chains $\overline{C}$ that are fixed by an automorphism f of G is $Fix_C(f) = \{C \in \overline{C} \mid f(C) = C\}$. From Burnside's Lemma, the number N of distinct fuzzy subgroups of G is given by

$$\frac{1}{|Aut(G)|}\sum_{f\in Aut(G)}|Fix_{\overline{C}}(f)|.$$

We remark that the formula above can be used to calculate the number N of DFS of any finite group provided the Lattice subgroup $L(G)$ and the automorphism group $Aut(G)$ of the group are known. For details on the new equivalence relation and the formula see Tarnauceanu (2016).

4. Distinct Fuzzy Subgroups of D_{2n}

In this section, we establish a bijective correspondence between the number of DFS of D_{2n} and the number of chains of sets subgroups fixed by a particular element $f_{\alpha,\beta} \in Aut(D_{2n})$. Furthermore, we derive a recurrence relation and generating function for counting DFS of D_{2n}. However, we begin by giving some properties of dihedral groups that are well known from literature.

4.1. *Properties of dihedral groups*

Dihedral group D_{2n} of order $2n$ with generators a,b for $n \geq 2$ has presentation given by

$$D_{2n} = \langle a,b \mid a^n = b^2 = 1, bab^{-1} = a^{-1} \rangle$$

From Tarnauceanu (2016), the automorphism group of D_{2n} is also well studied and is given by

$$Aut(D_{2n}) = \{ f_{\alpha,\beta} \mid \alpha = \overline{0,n-1},\ \beta = \overline{0,n-1} \}$$

where $\overline{0,n-1}$ are integers from 0 to n-1.

$$f_{\alpha,\beta} : D_{2n} \to D_{2n}$$

is defined by $f_{\alpha,\beta}(a^i) = a^{\alpha i}$ and $f_{\alpha,\beta}(a^i b) = a^{\alpha i+\beta} b, \forall\ i = \overline{0,n-1}$. The order of automorphism group of D_{2n} is given by; for $n \geq 2$, $|Aut(D_{2n})| = |n\varphi(n)|$. The structure of the subgroup lattice $L(D_{2n})$ of D_{2n} is also known: for every divisor r or n, D_{2n} possesses a subgroup isomorphic to Z_r, namely $H_0^r = \langle a^{n/r} \rangle$, and $\frac{n}{r}$ subgroups isomorphic to D_r, given by $f_{\alpha,\beta}(a^i) = a^{\alpha i} H_i^r = \langle a^{n/r}, a^{i-1}b \rangle, i = 1, 2, \cdots, \frac{n}{r}$. Now for each $f_{\alpha,\beta} \in Aut(D_{2n})$, let $Fix(f_{\alpha,\beta})$ be the set consisting of all subgroups of D_{2n} that are invariant relative to $f_{\alpha,\beta}$, that is

$$Fix(f_{\alpha,\beta}) = \{H \leq D_{2n} | f_{\alpha,\beta}(H) = H\}$$

A subgroup of type H_0^r belongs to $Fix(f_{\alpha,\beta})$ if and only if $(\alpha,r) = 1$, while a subgroup of type H_i^r belongs to $Fix(f_{\alpha,\beta})$ if and only if $(\alpha,r) = 1$ and $\frac{n}{r}$ divides $(\alpha - 1)(i - 1) + \beta$. Now computing $|Fix_C f_{\alpha,\beta}|$ implies counting the number of chains of $L(D_{2n})$ which ends in D_{2n} and are contained in the set $Fix(f_{\alpha,\beta})$.

4.2. *Recurrence relation for counting DFS of* D_{2n}

In establishing a bijective correspondence we consider many cases and apply the above properties.

4.2.1. *Case where n is a prime number p*

Observe that for any dihedral group D_{2p}, there exists an element $f_{2,0} \in Aut(D_{2p})$ such that,

$$Fix(f_{2,0}) = \{H_0^1, H_0^p, H_1^p, H_1^1\}, \text{ and } |Fix_C(f_{2,0})| = 6.$$

However, 6 corresponds to the number of DFS of D_{2p}. This shows that there is a bijective correspondence between the number of DFS of D_{2p} and $|Fix_C(f_{2,0})|$. It also shows that the number of DFS of D_{2p} is invariant with respect to the choice prime number.
We illustrate this case by using the following example;

Let $p = 3$, that is $D_{2\times3} = D_6$. We proceed as follows. $Aut(D_6) = \{f_{1,0}, f_{1,1}, f_{1,2}, f_{2,0}, f_{2,1}, f_{2,2}\}$. The sets of subgroups invariant with respect to each element of $Aut(D_{2p})$ are

Table 1: Subgroups fixed by $Aut(D_6)$

α	β	$Fix(f_{\alpha,\beta})$
1	0	{All lattice subgroups of D_6}
1	1,2	$\{H_0^1, H_0^3, H_1^3\}$
2	0	$\{H_0^1, H_0^3, H_1^3, H_1^1, H_1^3\}$
2	1	$\{H_0^1, H_0^3, H_3^1, H_1^3\}$
2	2	$\{H_0^1, H_0^3, H_2^1, H_1^3\}$

The following values are then obtained from the chains of subgroups fixed by each elements of the automorphism group:

- $|Fix_{\bar{C}}(f_{1,0})| = 10$
- $|Fix_{\bar{C}}(f_{1,1})| = |Fix_{\bar{C}}(f_{1,2})| = 4$
- $|Fix_{\bar{C}}(f_{2,0})| = |Fix_{\bar{C}}(f_{2,1})| = |Fix_{\bar{C}}(f_{2,2})| = 6$

The number A_k (k is number of distinct primes) of all distinct fuzzy subgroups of D_6 is;

$$A_1 = \frac{1}{6}[10 + (4 \times 2) + 6 + 6 + 6] = 6$$

Observe that $k = 1$ from the above example.

4.2.2 *Case where n is a product of two distinct primes*

In a similar manner we can establish a bijective correspondence between the number of DFS of D_{2n},where n is a product of two distinct primes, that is $D_{2p_1p_2}$ and $|Fix_C(f_{2,0})|$. Observe that for $D_{2p_1p_2}$, there exists an element $f_{2,0}$ $\in Aut(D_{2p_1p_2})$ such that $Fix(f_{2,0}) = \{H_0^1, H_0^{p_1}, H_0^{p_2}, H_1^{p_1}, H_1^{p_2}, H_0^{p_1p_2}, H_1^{p_1p_2}, H_1^1\}$, and $|Fix_C(f_{2,0})| = 26$. This value also corresponds to the number of DFS of $D_{2p_1p_2}$. This shows that there is a bijective correspondence between the number of DFS of $D_{2p_1p_2}$ and $|Fix_C(f_{2,0})|$. It shows that the number of DFS of $D_{2p_1p_2}$ increases as the number of prime numbers increases from one to two distinct primes.

To illustrate this case, we consider the following example. Suppose $p_1 = 5$, $p_2 = 3$ that is $D_{2\times5\times3} = D_{30}$. The order of the automorphism group of D_{30} is given by $|Aut(D_{30})| = 15 \times 8 = 120$. Next, we find the subgroups that are fixed by each automorphisms of D_{30}.

The subgroups of D_{30} are given below;

- $H_0^1 = \langle e \rangle$
- $H_0^k = \langle a^{\frac{15}{k}} \rangle$, for $k = 3, 5$ and 15.
- $H_k^1 = \langle a^{k-1}b \rangle$, for $k = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$ and 15
- $H_k^3 = \langle a^5, a^{k-1}b \rangle$ for $k = 1, 2, 3, 4$ and 5.

- $H_k^5 = \langle a^3, a^{k-1}b \rangle$ for $k = 1, 2$ and 3.
- $H_1^{15} = \langle a, b \rangle$.

The order of the automorphism group of D_{30} is given by $|Aut(D_{30})| = 15 \times 8 = 120$. Next we find the subgroups that are fixed by each automorphisms of D_{30}.

Table 2: Subgroups fixed by $Aut(D_{30})$

α	β	$Fix(f_{\alpha,\beta})$
1	0	{All lattice subgroups of D_{30}}
1	1,2,4,7,8,	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}\}$
	1,13,14	
1	3,6,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^5, H_2^5, H_3^5\}$
1	5,10	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3\}$
2,8,14	0	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^1, H_1^3, H_1^5\}$
2,8,14	1,7,13	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{15}^1, H_5^3, H_3^5\}$
2,8,14	2,14,11	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{14}^1, H_4^3, H_2^5\}$
2,8,14	3,6,9	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{13}^1, H_3^3, H_1^5\}$
2,8,14	4,13,7	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{12}^1, H_2^3, H_3^5\}$
2,8,14	5	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{11}^1, H_1^3, H_2^5\}$
2,8,14	6,12,3	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_{10}^1, H_5^3, H_1^5\}$
2,8,14	7,4,1	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_9^1, H_4^3, H_3^5\}$
2,8,14	8,11,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_8^1, H_3^3, H_2^5\}$
2,8,14	9,3,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_7^1, H_2^3, H_1^5\}$
2,8,14	10	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_6^1, H_1^3, H_3^5\}$
2,8,14	11,2,8,	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_5^1, H_5^3, H_2^5\}$
2,8,14	12,9,6	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_4^1, H_4^3, H_1^5\}$
2,8,14	13,1,5	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^1, H_3^3, H_3^5\}$
2,8,14	14,8,2	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_2^3, H_2^5\}$
4,7,13	0,0,0	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^1, H_6^1, H_{11}^1, H_1^3, H_1^5, H_2^5, H_3^5\}$
4,7,13	3,6,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_5^1, H_{10}^1, H_{15}^1, H_5^3, H_1^5, H_2^5, H_3^5\}$
4,7,13	6,12,9	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_4^1, H_9^1, H_{14}^1, H_4^3, H_1^5, H_2^5, H_3^5\}$
4,7,13	3,6,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_7^1, H_2^3, H_1^5\}$
4,7,13	6,12,9	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_6^1, H_1^3, H_3^5\}$

Table 2 *(Continued)*

Table 2 *(Continued)*

4,7,13	9,3,6	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^1, H_8^1, H_{13}^1, H_3^3, H_1^5, H_2^5, H_3^5\}$
4,7,13	12,9,3	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_2^3, H_1^5, H_2^5, H_3^5\}$
4,4,7,7,13,13	1,11,2,7,4,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_4^3\}$
4,4,7,7,13,13	2,7,4,14,8,13	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_2^3\}$
4,4,7,7,13,13	4,14,8,13,1,11	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_3^3\}$
4,4,7,7,13,13	5,10,5,10,5,10	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_1^3\}$
4,4,7,7,13,1	8,13,1,11,2,7	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_7^1, H_{12}^1, H_5^3\}$
11	0	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^1, H_4^1, H_7^1, H_{10}^1, H_{13}^1,$ $H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_1^5\}$
11	5	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^1, H_5^1, H_8^1, H_{11}^1,$ $H_{14}^1, H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_2^5\}$
11	10	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^1, H_6^1, H_9^1, H_{12}^1, H_{15}^1,$ $H_1^3, H_2^3, H_3^3, H_4^3, H_5^3, H_3^5\}$
11	1,4,7,13	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_3^5\}$
11	2,8,11,14	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_2^5\}$
11	3,6,9,12	$\{H_0^1, H_0^3, H_0^5, H_0^{15}, H_1^{15}, H_1^5\}$

Now, computing from subgroup lattice from Figure 2, we have the following values:

- $|Fix_{\bar{\mathcal{C}}}(f_{1,0})| = 134$
- $|Fix_{\bar{\mathcal{C}}}(f_{1,1})| = |Fix_{\bar{\mathcal{C}}}(f_{1,2})| = |Fix_{\bar{\mathcal{C}}}(f_{1,4})| = |Fix_{\bar{\mathcal{C}}}(f_{1,7})| = |Fix_{\bar{\mathcal{C}}}(f_{1,8})|$
  $= |Fix_{\bar{\mathcal{C}}}(f_{1,11})| = |Fix_{\bar{\mathcal{C}}}(f_{1,13})| = |Fix_{\bar{\mathcal{C}}}(f_{1,14})| = 12$
- $|Fix_{\bar{\mathcal{C}}}(f_{1,3})| = |Fix_{\bar{\mathcal{C}}}(f_{1,6})| = |Fix_{\bar{\mathcal{C}}}(f_{1,9})| = |Fix_{\bar{\mathcal{C}}}(f_{1,12})| = 24$
- $|Fix_{\bar{\mathcal{C}}}(f_{1,5})| = |Fix_{\bar{\mathcal{C}}}(f_{1,10})| = 32$
- $|Fix_{\bar{\mathcal{C}}}(f_{2,0})| = |Fix_{\bar{\mathcal{C}}}(f_{2,1})| = |Fix_{\bar{\mathcal{C}}}(f_{2,2})| = |Fix_{\bar{\mathcal{C}}}(f_{2,3})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{2,4})| = |Fix_{\bar{\mathcal{C}}}(f_{2,5})| = |Fix_{\bar{\mathcal{C}}}(f_{2,6})| = |Fix_{\bar{\mathcal{C}}}(f_{2,7})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{2,8})| = |Fix_{\bar{\mathcal{C}}}(f_{2,9})| = |Fix_{\bar{\mathcal{C}}}(f_{2,10})| = |Fix_{\bar{\mathcal{C}}}(f_{2,11})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{2,12})| = |Fix_{\bar{\mathcal{C}}}(f_{2,13})| = |Fix_{\bar{\mathcal{C}}}(f_{2,14})| = |Fix_{\bar{\mathcal{C}}}(f_{8,0})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{8,1})| = |Fix_{\bar{\mathcal{C}}}(f_{8,2})| = |Fix_{\bar{\mathcal{C}}}(f_{8,3})| = |Fix_{\bar{\mathcal{C}}}(f_{8,4})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{8,5})| = |Fix_{\bar{\mathcal{C}}}(f_{8,6})| = |Fix_{\bar{\mathcal{C}}}(f_{8,7})| = |Fix_{\bar{\mathcal{C}}}(f_{8,8})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{8,9})| = |Fix_{\bar{\mathcal{C}}}(f_{8,10})| = |Fix_{\bar{\mathcal{C}}}(f_{8,11})| = |Fix_{\bar{\mathcal{C}}}(f_{8,12})| =$
  $|Fix_{\bar{\mathcal{C}}}(f_{8,13})| = |Fix_{\bar{\mathcal{C}}}(f_{8,14})| = |Fix_{\bar{\mathcal{C}}}(f_{14,0})| = |Fix_{\bar{\mathcal{C}}}(f_{14,1})| =$

$$
\begin{aligned}
&|Fix_{\bar{C}}(f_{14,2})|=|Fix_{\bar{C}}(f_{14,3})|=|Fix_{\bar{C}}(f_{14,4})|=|Fix_{\bar{C}}(f_{14,5})|=\\
&|Fix_{\bar{C}}(f_{14,6})|=|Fix_{\bar{C}}(f_{14,7})|=|Fix_{\bar{C}}(f_{14,8})|=|Fix_{\bar{C}}(f_{14,9})|=\\
&|Fix_{\bar{C}}(f_{14,10})|=|Fix_{\bar{C}}(f_{14,11})|=|Fix_{\bar{C}}(f_{14,12})|=|Fix_{\bar{C}}(f_{14,13})|\\
&=|Fix_{\bar{C}}(f_{14,14})|=26
\end{aligned}
$$

- $$
\begin{aligned}
&|Fix_{\bar{C}}(f_{4,0})|=|Fix_{\bar{C}}(f_{4,3})|=|Fix_{\bar{C}}(f_{4,6})|=|Fix_{\bar{C}}(f_{4,9})|=|Fix_{\bar{C}}(f_{4,12})|=|Fix_{\bar{C}}(f_{7,0})|\\
&=|Fix_{\bar{C}}(f_{7,3})|=|Fix_{\bar{C}}(f_{7,6})|=|Fix_{\bar{C}}(f_{7,9})|=|Fix_{\bar{C}}(f_{7,12})|=|Fix_{\bar{C}}(f_{13,0})|=|Fix_{\bar{C}}(f_{13,3})|\\
&=|Fix_{\bar{C}}(f_{13,6})|=|Fix_{\bar{C}}(f_{13,9})|=|Fix_{\bar{C}}(f_{13,12})|=46
\end{aligned}
$$

- $|Fix_{\bar{C}}(f_{11,0})|=|Fix_{\bar{C}}(f_{11,5})|=|Fix_{\bar{C}}(f_{11,10})|=66$
- $$
\begin{aligned}
&|Fix_{\bar{C}}(f_{4,1})|=|Fix_{\bar{C}}(f_{4,2})|=|Fix_{\bar{C}}(f_{4,4})|=|Fix_{\bar{C}}(f_{4,5})|=|Fix_{\bar{C}}(f_{4,7})|=|Fix_{\bar{C}}(f_{4,8})|=\\
&|Fix_{\bar{C}}(f_{4,10})|=|Fix_{\bar{C}}(f_{4,11})|=|Fix_{\bar{C}}(f_{4,13})|=|Fix_{\bar{C}}(f_{4,14})|=|Fix_{\bar{C}}(f_{7,1})|=|Fix_{\bar{C}}(f_{7,2})|=\\
&|Fix_{\bar{C}}(f_{7,4})|=|Fix_{\bar{C}}(f_{7,5})|=|Fix_{\bar{C}}(f_{7,7})|=|Fix_{\bar{C}}(f_{7,8})|=|Fix_{\bar{C}}(f_{7,10})|=|Fix_{\bar{C}}(f_{7,11})|=\\
&|Fix_{\bar{C}}(f_{7,13})|=|Fix_{\bar{C}}(f_{7,14})|=|Fix_{\bar{C}}(f_{11,1})|=|Fix_{\bar{C}}(f_{11,2})|=|Fix_{\bar{C}}(f_{11,3})|=|Fix_{\bar{C}}(f_{11,4})|=\\
&|Fix_{\bar{C}}(f_{11,6})|=|Fix_{\bar{C}}(f_{11,7})|=|Fix_{\bar{C}}(f_{11,8})|=|Fix_{\bar{C}}(f_{11,9})|=|Fix_{\bar{C}}(f_{11,11})|=|Fix_{\bar{C}}(f_{11,12})|=\\
&|Fix_{\bar{C}}(f_{11,13})|=|Fix_{\bar{C}}(f_{11,14})|=|Fix_{\bar{C}}(f_{13,1})|=|Fix_{\bar{C}}(f_{13,2})|=|Fix_{\bar{C}}(f_{13,4})|=|Fix_{\bar{C}}(f_{13,5})|=\\
&|Fix_{\bar{C}}(f_{13,7})|=|Fix_{\bar{C}}(f_{13,8})|=|Fix_{\bar{C}}(f_{13,10})|=|Fix_{\bar{C}}(f_{13,11})|=|Fix_{\bar{C}}(f_{13,13})|=|Fix_{\bar{C}}(f_{13,14})|=16
\end{aligned}
$$

From the computation above we have that; the number A_k of distinct fuzzy subgroups of D_{30} with respect to $\approx$ is given by the equality;

$$
\begin{aligned}
A_2 &= \frac{1}{120}[134+(12\times 8)+(24\times 4)+(32\times 2)+(26\times 3\times 15)+(46\times 3\times 5)+(16\times 3\times 10)\\
&+(66\times 3)+(16\times 12)]\\
&=26.
\end{aligned}
$$

Notice that to obtain $f_{2,0}\in Aut(D_{2p_1p_2})$, we set $p_1, p_2 \neq 2$. However, suppose $p_1 = 2$ or $p_2 = 2$, our result will remain valid since the number of DFS is invariant for any two distinct primes.

Table 3: The number DFS of D_{2n} for a few number of distinct primes

Group D_{2n}	$D_{2p_1p_2}$	$D_{2p_1p_2p_3}$	$D_{2p_1p_2p_3p_4}$	$D_{2p_1p_2p_3p_4p_5}$
A_k	26	150	1082	9366
k	2	3	4	5

[a]Whenever $k = 3$, then $A_k = 150$, for details, see Olayiwola and Isyaku (2018). Similar procedure also applies for $k = 4, 5$.

These numbers correspond to two times the ordered Bell numbers. The ordered Bell numbers are used to represent the distinct rational preferential arrangement available to a person faced with k distinguishable decisions allowing indifference, see Gross (1962) and Murali (2006). These numbers also form a sequence with ID Number A000670 on Online Encyclopedia of Integer Sequence.

More generally, we conclude that there is bijective correspondence between the number of DFS of D_{2n} and chains of subgroups of D_{2n} fixed by the element of automorphism group $f_{2,0}$. Also, the number of DFS of D_{2n} is invariant with respect to the choice distinct primes.

To establish the recurrence relation for the number of DFS of D_{2n}, we assume an initial condition for our recurrence relation. The initial condition specifies the terms before recurrence relation takes effect. Set $A_0 = 1$ as initial condition. The case $k = 1,2$ had earlier been computed, while the case $k = 3$ was computed by Olayiwola and Isyaku (2018), so $A_1 = 6$, $A_2 = 26$ and $A_3 = 150$. We can write A_k in terms of ordered Bell number, B_k. These Bell numbers can be computed in the following manner:

$$B_n = \sum_{i=0}^{n-1} \binom{n}{i} b_i \quad \text{with } B_0 = 1.$$

Table 4: The values of A_k is presented in a Pascal like triangle as follows:

2	2	1				
2	3(3)	3	1			
2	13(4)	3(6)	4	1		
2	75(5)	13(10)	3(10)	5	1	
2	541(6)	75(15)	13(20)	3(15)	6	1

The first and last entry in each row are fixed, they are 2 and 1 respectively. Entries in brackets are consistent with the entries obtained in classical Pascal triangle. Entries multiplying the brackets on the second column of each row are obtained by adding only the entries of the preceding row without the fixed 2. For example, the first row is A_1 and we have $3 = (2 + 1) = 3$. Also, the second row A_2, we have $13 = 3(3) + 3 + 1 = 13$. We can now compute $A_1 = 2 \times (2 + 1) = 6$, $A_2 = 2 \times [3(3) + 3 + 1] = 26$. The following lemma gives the relationship between A_k and B_k. We state without proof as the proof is straightforward.

Lemma 4.3. *The number of distinct fuzzy subgroups of D_{2n}, is given by $A_k = 2B_{k+1}$. $A_0 = 1$ and $k \geq 1$.*

4.3.1 *Exponential Generating Function for A_k*

In what follows, we derive the exponential generating function for A_k.

Theorem 4.4. The exponential generating function of $A_k = 2B_{k+1}$ *is*

$$A_k = \frac{2e^x - (2-e^x)^2}{(2-e^x)^2}.$$

Proof: According to Trojovsky (2006), the exponential generating function for B_k is given by $B_E(x) = \dfrac{1}{2-e^x}$. If $B_E(x)$ is exponential generating function of a sequence B_k, then the derivative of $B_E(x)$, that is $B_E'(x) = \dfrac{e^x}{(2-e^x)^2}$ gives the exponential generating function of B_{k+1}, see Wilf (1994). Let

$$\frac{e^x}{(2-e^x)^2} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{k!} x^k,$$

then

$$\frac{e^x}{(2-e^x)^2} = b_0 + \sum_{k=1}^{\infty} \frac{b_{k+1}}{k!} x^k$$

$$\frac{e^x}{(2-e^x)^2} = b_0 + \frac{1}{2}\sum_{k=0}^{\infty} \frac{2b_{k+1}}{k!} x^k.$$

But $A_k = 2B_{k+1}$, thus

$$\frac{e^x}{(2-e^x)^2} = b_0 + \frac{1}{2}\sum_{k=1}^{\infty} \frac{a_k}{k!} x^k.$$

But

$$\frac{1}{2}\sum_{k=1}^{\infty} \frac{a_k}{k!} x^k = \frac{1}{2}\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k - \frac{1}{2}a_0.$$

Thus,

$$\frac{e^x}{(2-e^x)^2} = b_0 + \frac{1}{2}\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k - \frac{1}{2}a_0.$$

Recall that, $b_0 = 1 = a_0$, hence

$$\frac{e^x}{(2-e^x)^2} = a_0 + \frac{1}{2}\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k - \frac{1}{2}a_0,$$

then we have

$$\frac{e^x}{(2-e^x)^2} = \frac{1}{2}\sum_{k=0}^{\infty}\frac{a_k}{k!}x^k + \frac{1}{2}$$

and finally,

$$\frac{2e^x-(2-e^x)^2}{(2-e^x)^2} = \sum_{k=0}^{\infty}\frac{a_k}{k!}x^k.$$

Hence the generating function for A_k is

$$A_E(x) = \frac{2e^x-(2-e^x)^2}{(2-e^x)^2} = 1+6x+\frac{26x^2}{2!}+\frac{150x^3}{3!}+\frac{1082x^4}{4!}+\cdots \square$$

The k-th term of A_k are the coefficients of $\dfrac{x^k}{k!}$ in the expansion above.

5. Conclusion

For DFS of dihedral group D_{2n}, where n is a product of finite distinct number of prime number, we have established the following results:

- The number of DFS of D_{2n} is invariant with respect to the choice of distinct primes but varies with respect to the number of distinct primes.
- Counting DFS of D_{2n} has been sufficiently reduced to counting the chains of subgroups fixed by $f_{2,0}$.
- The number of DFS of D_{2n} can be computed from the recurrence relation $A_k = 2B_{k+1}$ with $A_0 = 1$.
- The generating function for DFS of D_{2n} is given by

$$A_E(x) = \frac{2e^x-(2-e^x)^2}{(2-e^x)^2}.$$

We hope that this results will serve as a motivation towards establishing some explicit and more efficient formulas for counting DFS of other well known finite groups.

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