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On the Stability and Ultimate Boundedness of Solutions for Certain Third Order Differential Equations

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Abstract: Problem Statement: With respect to our observation in the relevant literature, work on stability and boundedness of solution for certain third order nonlinear differential equations where the nonlinear and the forcing terms depend on certain variables are scare. The objective of this study was to get criteria for stability and boundedness of solutions for these classes of differential equations. **Approach:** Using Lyapunov second or direct method, a complete Lyapunov function was constructed and used to obtain our results. **Results:** Conditions were obtained for: (i) Uniform asymptotic stability and, (ii) Uniform ultimate boundedness, of solutions for certain third order non-linear non-autonomous differential equations. **Conclusion:** Our results do not only bridge the gap but extend some well known results in the literature.

Key words: Asymptotic stability, uniform ultimate boundedness; third order; complete Lyapunov function.

INTRODUCTION

We shall be concerned here, with uniform asymptotic stability of the zero solutions (that is when p(t, x y, z) = 0) and uniform ultimate boundedness of solutions of the third order, non-linear, non-autonomous differential equations:

$$\dddot{x} + f(t,x,\dot{x})\ddot{x} + q(t)g(\dot{x}) + r(t)h(x) = p(t,x,\dot{x},\ddot{x}) \qquad (1)$$

On setting $\dot{x} = y$, $\ddot{x} = z$ Eq. 1 is equivalent to the system of differential equation:

$$\dot{x} = y,\ \dot{y} = z,$$
$$\dot{z} = p(t,x,y,z) - f(t,x,y)z - q(t)g(y) - r(t)h(x) \qquad (2)$$

In which:

$$p:\mathbb{R}^{+}\times\mathbb{R}^{3}\to\mathbb{R};\ F:\mathbb{R}^{+}\times\mathbb{R}^{2}\to\mathbb{R};\ g,h:\mathbb{R}\to\mathbb{R};\ q,r:\mathbb{R}^{+}\to\mathbb{R};$$
$$\mathbb{R} = (-\infty,\infty);\ \mathbb{R}^{+} = [0,\infty);$$

p, f, g, h, q and r depend only on the arguments displayed explicitly and $\frac{\partial}{\partial t}f(t,x,y) = f_t(t,x,y)$, $\frac{\partial}{\partial x}f(t,x,y) = f_x(t,x,y)$, $\frac{d}{dx}h(x) = h'(x)$, $\frac{d}{dt}q(t) = q'(t)$ and $\frac{d}{dt}r(t) = r'(t)$ exist and are continuous for all t, x, and y. The dots here as elsewhere, stand for differentiation with respect to the independent variable t. Moreover, the existence and uniqueness of solutions of (1) will be assumed. Stability analysis and ultimate boundedness of solutions of nonlinear systems are important area of current research and many concept of stability boundedness of solutions have in the past and also recently been studied, see for instance[14], a survey book, Rouche *et al.*[15] and Yoshizawa[21, 22] are background books. The studies of qualitative behaviour of solutions have been discussed by many authors in a series of research study. See for instance[1-13,16-20] and references therein. These study were done with the aid of Lyapunov functions except in[2, 3] where frequency domain approaches were used. With respect to our observation in the relevant literature, these authors considered stability, asymptotic behaviour, boundedness of solutions of Eq. 1, 2 in the case $f(t,x,\dot{x})$ equal any of $f(x,\dot{x},\ddot{x}), f(x,\dot{x}), f(x)$ and a where a is positive constant and q(t) = r(t) = 1.

In[17] Swick discussed conditions for uniform boundedness of Eq. 1 when $p(t,x,\dot{x},\ddot{x}) \equiv 0$ using an incomplete Lyapunov functions.

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MATERIALS AND METHODS

In this study, conditions for uniform asymptotic stability and uniform ultimate boundedness of solutions of the nonlinear differential Eq. 1 will be considered with the aid of an effective method for studying stability and ultimate boundedness of solutions namely Lyapunov second or direct method. Here a complete Lyapunov function was constructed and used to obtain the following results.

RESULTS

In the case p(t, x, y, z) = 0 (1) and its equivalent system(2) become

$$\dddot{x}+f(t,x,\dot{x})\ddot{x}+q(t)g(\dot{x})+r(t)h(x)=0 \qquad (3)$$

and

$$\begin{cases}\dot{x}=y,\dot{y}=z\\ \dot{z}=-f(t,x,y)z-q(t)g(y)-r(t)h(x)\end{cases} \qquad (4)$$

with the following result

Theorem 1: In addition to the basic assumptions on the functions f,g,h,q and r, suppose that $\alpha,\alpha_1,b,b_1,c,d,d_1$, are positive constants and for all t = 0

- $h(0)=0, \frac{h(x)}{x}\geq\delta_0$ for all x?0;
- $h'(x) = c$ for all x;
- $b\leq\frac{g(y)}{y}\leq b_1$ for all y?0;
- $d_1=r(t) = q(t), q'(t) = r'(t) = 0$
- $a=f(t, x, y) = a_1$ for all x and y
- $yf_x(t, x, y) = 0, f_t(t, x, y) = 0$ for all x and y.

Then the zero solution of 4 is uniform asymptotically stable.

In the case $p(t,x,\dot{x},\ddot{x})=p(t)\neq 0$ Eq. (1) and (2) become

$$\dddot{x}+f(t,x,\dot{x})+q(t)g(\dot{x})+r(t)h(x)=p(t) \qquad (5)$$

and

$$\begin{cases}\dot{x}=y,\dot{y}=z\\ \dot{z}=p(t)-f(t,x,y)z-q(t)g(y)-r(t)h(x)\end{cases} \qquad (6)$$

with the following statement:

Theorem 2: Suppose that: (i) hypotheses (i)-(iv) of Theorem 1 hold;

- $|p(t)|\leq P_0<\infty, P_0\geq 0 \forall\ t\geq 0.$

Then the solution (x(t),y(t),z(t)) of (6) is uniformly ultimately bounded.

Theorem 3: Suppose that: (i) hypothesis (i) of Theorem 2 holds:

- $|p(t,x,y,z)|\leq p_1(t)+p_2(t)(|x|+|y|+|z|)$

provided that $|x|+|y|+|z|\geq p, 0\leq p<\infty$, where $p_1(t)$ and $p_2(t)$ are non-negative continuous functions satisfying:

$$p_1(t)\leq P_1, 0\leq P_1<8,\ \forall t\geq 0 \qquad (7)$$

and there is $\in\ > 0$ such that if:

$$0\leq p_2(t)\leq\in\quad t\geq 0. \qquad (8)$$

Then the solution (x(t),y(t), z(t)) of (2) is uniformly ultimately bounded.

DISCUSSION

The proofs of Theorem 1, 2 and 3 depend on the continuously differentiable function V = V(t, x, y, z) defined by:

$$\begin{aligned}2V &= 2(\alpha+a)r(t)H(x)+4q(t)G(y)+4r(t)h(x)y\\ &+2z^2+\beta y^2+b\beta x^2+2a\beta xy+2\beta xz+2(\alpha+a)yz\\ &+2(\alpha+a)\int_0^y \tau f(t,x,\tau)d\tau\end{aligned} \qquad (9)$$

where α and β are positive constants satisfying:

$$\frac{c}{b}<\alpha<a \qquad (10a)$$

and

$$o < \beta < \min\{b - \frac{c}{a};$$

$$\delta_1 (ab - c)\left\{a + 1 + \delta_0^{-1}\delta_1^{-1}\left[q(t)\frac{g(y)}{y} - b\right]^2\right\}^{-1}; \qquad (10b)$$

$$(a - \alpha)\, 2^{-1}\left[a + 1 + \delta_0^{-1}\delta_1^{-1}[f(t,x,y) - a]^2\right]^{-1}$$

and that:

$$H(x) = \int_0^x h(\xi)d\xi \text{ and } G(y) = \int_0^y g(\tau)d\tau$$

Lemma 1: Subject to conditions (i)-(iv) of Theorem 1, V(t,0,0,0) = 0 and there exist positive constants $D_0 = D_0(\alpha, b, c, d_0, d_1\, \alpha, \beta)$ and $D_1 = D_1(\alpha, \alpha_1, b, b_1, c, r_0, q_o, \alpha, \beta)$ such that:

$$D_0(x^2(t)+y^2(t)+z^2(t)) = V(t) = D_1(x^2(t)+y^2(t)+z^2(t)) \qquad (11)$$

Proof: Setting x(t) = y (t) = z (t) = 0 in (9), clearly V(t,0,0,0) = 0. Then (9) can be recast in the form:

$$2V = V_1 + V_2 + V_3 \qquad (12a)$$

Where

$$V_1 = 2(\alpha+a)\, r(t)\, H(x) + 4q(t)\, G(y) + 4r(t)\, h(x)y, \qquad (12b)$$

$$V_2 = b\beta x^2 + (\alpha^2+ß)y^2 + z^2 + 2\alpha ß xy + 2ß xz + 2ayz, \qquad (12c)$$

and

$$V_3 = z^2 + 2\alpha yz + \alpha^2 y^2 + 2(\alpha + a)\int_0^y \tau f(t,x,\tau)d\tau - (\alpha^2 + a^2)y^2. \qquad (12d)$$

In view of hypothesis (iv) of the Theorem 1, $r(t) \ge d_1$ and $q(t) \ge r(t)$ together imply:

$$\frac{1}{2}V_1 \ge \delta_1[(\alpha + a)H(x) + 2G(y) + 2h(x)y] \qquad (13)$$

By hypothesis (ii) of Theorem 1, we have:

$$2G(y) + 2h(x)\, y \ge -b^{-1}h(x) \qquad (14a)$$

since b>0 and $(by+h(x))^2 \ge 0$ for all x and y . Also from hypotheses (i) and (ii) and the fact that:

$$h^2(x) = 2\int_0^x h'(\xi)h(\xi)d\xi$$

Since h(0) = 0, we have:

$$(\alpha + a)H(x) \ge \frac{1}{2}[(\alpha+\alpha)b - 2c]b^{-1}\delta_0 x^2 + b^{-1}h^2(x) \qquad (14b)$$

On gathering (14a) and (14b) into (13), we obtain:

$$V_1 \ge [(\alpha+a)b - 2c]b^{-1}d_0\, d_1\, x^2 \qquad (15a)$$

Also V_2 can be recast in the form XPX^T, where:

$$X = (xyz), P = \begin{pmatrix} b\beta & \alpha\beta & \beta \\ \alpha\beta & (\alpha^2+\beta) & a \\ \beta & a & 1 \end{pmatrix}$$

and X^T is the transpose of X. The eigenvalues of matrix P will all be positive, thus det P = $ß^2(b - ß) > ß^2$ since $b - ß > 0$ by (10b), so that:

$$V_2 \ge ß^2(x^2+y^2+z^2) \qquad (15b)$$

Finally, by hypothesis (v) of the theorem $f(t,x,y) \ge a$ for all x,y and $t \ge 0$ then:

$$V_3 \ge \alpha y^2, \qquad (15c)$$

since α-a > 0 by (10a) and $(z + \alpha y)^2 \ge 0$ for all y and z. A combination of estimates (15a), (15b) and (15c) yields :

$$V \ge \frac{1}{2}\left[[(\alpha+a)b - 2c]b^{-1}\delta_0\delta_1 + \frac{1}{2}\beta^2\right]x^2 + \frac{1}{2}(\alpha+\beta^2)y^2 + \frac{1}{2}\beta^2 z^2.$$

By (10) $\alpha b - c > 0$, ab-c > 0 and a, b, d_0, d_1, α, ß are all positive constants, there exists a positive constant:

$$\delta_2 = \frac{1}{2}\min\left[[(\alpha+a)b - 2c]b^{-1}\delta_0\delta_1 + \frac{1}{2}\beta^2; (\alpha+\beta^2); \beta^2\right]$$

such that:

$$V \ge d_2(x^2+y^2+z^2) \qquad (16)$$

for all x, y, z and $t \ge 0$, this established the lower inequality in (11). To prove the upper inequality in (11), hypotheses (ii) and (iv) of the theorem imply that:

$$h(x) \le c\,x \text{ for all } x \ne 0 \qquad (17a)$$

and

$r(t) \le r_0$ and $q(t) \le q_0$ for $t \ge 0$ (17b)

where r_0 and q_0 are positive constants.
From estimates (17a) and (17b), we obtain:

$$V_1 \le \frac{1}{2}\left(\alpha+a+\frac{1}{2}\right)c r_0 x^2 + (b_1 q_0 + c r_0)y^2 \quad (18a)$$

where we have used the inequality:

$$2|x||y| \le x^2 + y^2$$

Sum of (12c) and (12d) together with the Young's inequality yields:

$$V_2+V_3 \le (a+b+1)\beta x^2 + [\beta(a+1)+(\alpha+a)(a_1+1)]y^2 + (\alpha+\beta+a+2)z^2 \quad (18b)$$

Substituting (18a) and (18b) into (9) to get:

$$V \le \frac{1}{2}\left[(a+b+1)+\frac{1}{2}\left(\alpha+a+\frac{1}{2}\right)c r_0\right]x^2 + \frac{1}{2}[(\alpha+a)(a_1+1) + \beta(a+1)+(b_1 q_0 + c r_0)]y^2 + \frac{1}{2}(\alpha+\beta+a+2)z^2$$

Since a,b,c,a,ß,a_1,b_1,r_0 and q_0 are positive constants, there exists a positive constant:

$$\delta_3 = \frac{1}{2}\max\left[\left[(a+b+1)+\frac{1}{2}\left(\alpha+a+\frac{1}{2}\right)c r_0\right]; [(\alpha+a)(a_1+1)+\beta(a+1)+b_1 q_0 + c r_0)]; (\alpha+\beta+a+2)\right]$$

such that:

$$V \le d_3(x^2+y^2+z^2) \quad (19)$$

Equation 19 is the upper inequality in (11), and hence estimate (16) clearly implies that V(t, x, y, z) ? + 8 as $(x^2+y^2+z^2)$? 8 . From (16) and (19), Lemma 1 is established.

Lemma 2: Under the hypotheses of Theorem 1, there exists a constant $D_3>0$ depending only on a,b,c, d_0, d_1, α and β such that if (x(t), y(t), z(t)) is any solution of (4), then:

$$\dot{V} \equiv \frac{d}{dt}V(t,x(t),y(t),z(t)) \le -D_3(x^2(t)+y^2(t)+z^2(t)) \quad (20)$$

Proof: Let (x(t),y(t),z(t)) be any solution of (4), then an elementary calculation of (12), and (4) yields:

$$\begin{aligned}\dot{V}_{(4)} &= (\alpha+a)r'(t)H(x)+2q'(t)G(y)+2r'(t)yh(x)-\beta r(t)h(x)x \\ &+(\alpha+a)\int_0^y \tau f_t(t,x,\tau)d\tau + (\alpha+a)y\int_0^y \tau f_x(t,x,\tau)d\tau + a\beta y^2 \\ &-r(t)\left[(\alpha+a)\frac{q(t)g(y)}{r(t)y} - 2h'(x)\right]y^2 - \beta[f(t,x,y)-a]xz \\ &-[2f(t,x,y)-(\alpha+a)]z^2 - \beta\left[\frac{q(t)g(y)}{y}-b\right]xy + 2\beta yz.\end{aligned}$$

In view of hypothesis (vi) of Theorem 1,

$$(\alpha+a)\left[\int_0^y \tau f_t(t,x,\tau)d\tau + y\int_0^y \tau f_x(t,x,\tau)d\tau\right] \le 0,$$

for all x, y and t ≥ 0 since α and a are positive constants and by Young's inequality, we have:

$$\begin{aligned}\dot{V}_{(4)} &\le W_1 - W_2 + (a+1)\beta y^2 + \beta z^2 \\ &-\beta[f(t,x,y)-a]xz - \beta\left[\frac{q(t)g(y)}{y}-b\right]xy\end{aligned} \quad (21)$$

Where:

$W_1 = (\alpha+a)r'(t)H(x) + 2q'(t)G(y) + 2r'(t)yh(x)$

and

$$W_2 = \beta r(t)h(x)x + r(t)\left[(\alpha+a)\frac{q(t)g(y)}{r(t)y} - 2h'(x)\right]y^2 + [2f(t,x,y)-(\alpha+a)]z^2$$

Now if $r'(t)=0$ for such t's we have:

$W_1=2q'(t)G(y)$

then by condition (ii) $G(y)\ge 0$ for all y?0, so that $W_1\le 0$ since $q'(t) \le 0$ for all $t \ge 0$.

If $r'(t) < 0$, since $q'(t) \le r'(t)$ it follows that:

$W_1 \le r'(t)[(\alpha+a)H(x)+2G(y)+2yh(x)]$.

It is clear, from (14a) and (14b) that:

$(\alpha+a)H(x)+2G(y)+2yh(x) \ge 0$

for all x and y, since α and a are positive constants, hence in both cases we have:

$$W_1 \le 0 \tag{22}$$

Also hypotheses (i) and (iv) of Theorem 1, imply

$$ßr(t)h(x)x \ge ßd_0ß_1x^2 \ \forall x \ne 0 \tag{23a}$$

Since $r(t) \le q(t)$, $g(y) \ge by$ $(y \ne 0)$, $h'(x) \le c \,\forall x$ and $r(t) \ge d_1 \forall t \ge 0$, it follows by (10) that:

$$r(t)\left[(\alpha+a)\frac{q(t)g(y)}{r(t)y} - 2h'(x)\right] \ge \delta_1[(\alpha+a)b-2c] > 0 \tag{23b}$$

By hypothesis (v) $f(t,x,y) \ge a$ for all x,y and $t \ge 0$ so that:

$$2f(t,x,y) - (\alpha+a) \ge a-\alpha \tag{23c}$$

On gathering estimates (23a), (23b) and (23c), we obtain:

$$W_2 \ge ßd_0d_1x^2+d_1[(\alpha+a)b-2c]y^2+(a-\alpha)z^2 \tag{24}$$

From (22) and (24), estimate (21) becomes:

$$\dot{V}_4 \le -\beta\,\delta_0\delta_1x^2 - \delta_1(\alpha b - c)y^2 - [\delta_1(ab-c)$$
$$-(a+1)\beta]y^2 - \frac{1}{2}(a-\alpha)z^2 - \left[\frac{1}{2}(a-\alpha)-\beta\right]z^2$$
$$-\beta\left[\frac{q(t)g(y)}{y} - b]\right]xy - \beta[f(t,x,y)-a]xz$$
$$\dot{V}_4 \le -\frac{1}{2}\beta\,\delta_0\delta_1x^2 - \delta_1(\alpha b - c)y^2 - \frac{1}{2}(a-\alpha)z^2$$
$$-\left[\delta_1(ab-c) - \beta\left[a+1+\delta_0^{-1}\delta_1^{-1}\left[\frac{q(t)g(y)}{y}-b\right]^2\right]\right]y^2$$
$$-\left[\frac{1}{2}(a-\alpha) - \beta[1+\delta_0^{-1}\delta_1^{-1}[f(t,x,y)-a]^2]\right]z^2$$
$$-\frac{1}{4}\beta\delta_0\delta_1\left[x+2\delta_0^{-1}\delta_1^{-1}\left[[\frac{q(t)g(y)}{y}-b]\right]y\right]^2$$
$$-\frac{1}{4}\beta\delta_0\delta_1[x+2\delta_0^{-1}\delta_1^{-1}[f(t,x,y)-a]z]^2$$

Now by (10) and the fact that:

$$\left[x+2\delta_0^{-1}\delta_1^{-1}\left[\frac{q(t)g(y)}{y}-b\right]y\right]^2 \ge 0$$
$$[x+2\delta_0^{-1}\delta_1^{-1}[f(t,x,y)-a]z]^2 \ge 0$$

for all x,y,z and $t \ge 0$ and $ß, d_0$ d_1 are all positive constants, there exists a positive constant:

$$\delta_4 = \min\left[\frac{1}{2}\beta\,\delta_0\delta_1;\delta_1(\alpha b - c);\frac{1}{2}(a-\alpha)]\right]$$

such that for all x,y,z and $t \ge 0$:

$$\dot{V}_{(4)} \le -d_4(x^2+y^2+z^2) \tag{25}$$

This completes the proof of Lemma 2.

Proof of Theorem 1: Let (x(t),y(t),z(t)) be any solution of (4). To prove the Theorem 1, we shall use the usual limit point argument as is contained in [23] to show that when Lemma 1 and Lemma 2 hold, then V(t) $V(t) \equiv V(t,x(t),y(t),z(t)) \to 0$ as $t \to 8$. In view of the fact that from Lemma 1 V (t,x,y,z) = 0 if and only if $x^2+y^2+z^2 = 0$, V(t,x,y,z) > 0 if and only if $x^2+y^2+z^2 \ne 0$, V (t,x,y,z) $\to +8$ if and only if $x^2+y^2+z^2 \to 8$. The remaining of this proof follows the strategy indicated in[11], and hence it omitted. This completes the proof of Theorem 1.

Proof of Theorem 2: Let (x(t),y(t),z(t) be any solution of (6). According to Lemma 1 and Lemma 2, it follows that V(t,x,y,z) = 0 if and if only if $x^2 + y^2 + z^2 = 0$, V(t,x,y,z) > 0 if and if only if $x^2 + y^2 + z^2 \ne 0$, V(t,x,y,z) $\to +8$ if and if only if $x^2 + y^2 + z^2 \to 8$. Along any solution (x(t),z(t),z(t)) we have:

$$\dot{V}_{(6)} = \dot{V}_{(4)} + \beta x + (\alpha + a)y + 2z]p(t)$$

By Lemma 2:

$$\dot{V}_{(4)} \le -\delta_4(x^2+y^2+z^2)$$

in (25), so that:

$$\dot{V}_{(6)} \le -\delta_4(x^2+y^2+z^2) + 3^{1/2}\delta_5P_0(x^2+y^2+z^2)^{1/2} \tag{26}$$

where $d_5 = \max[ß;\alpha+a;2]$. Choose:

$$\left(x^2+y^2+z^2\right)^{\frac{1}{2}} \ge 2\sqrt{3}\delta_5\delta_4^{-1} = \delta_6$$

the inequality in (26) becomes:

$$\dot{V}_{(6)} \le -\frac{1}{2}\delta_4(x^2+y^2+z^2) \le -\delta_7 \tag{27}$$

provided that:

$$x^2 + y^2 + z^2 \geq 2\delta_7\delta_4^{-1} = \delta_8$$

Theorem 2 follows from (16), (19) and (27), see for instance[22].

Proof of Theorem 3: Along any solution (x(t),y(t), z(t)) we have:

$$\dot{V}_{(2)} = \dot{V}_{(4)} + [\beta x + (\alpha + a)y + 2z]p(t,x,y,z)$$

By Lemma 2, $\dot{V}_{(4)} \leq -\delta_4\left(x^2 + y^2 + z^2\right)$ in (25), and by condition (ii) of Theorem 3, we have:

$$\dot{V}_{(2)} \leq \dot{V}_{(4)} + \delta_5(|x| + |y| + |z|)[p_1(t) + p_2(t)(|x| + |y| + |z|)]$$
$$\leq -(\delta_4 - 3\delta_5 \in)(x^2 + y^2 + z^2) + 3^{1/2}\delta_5(x^2 + y^2 + z^2)^{1/2}$$

Choose $\in$ so small so that $d_4 > 3\, d_5 \in$, there exist positive constants d_9 and d_{10} such that:

$$\dot{V}_{(2)} \leq -\delta_9\left(x_2 + y^2 + z^2\right) + \delta_{10}\left(x^2 + y^2 + z^2\right)^{1/2}.$$

The remaining of this proof follows the strategy indicated in the proof of Theorem 2. Hence it is omitted; this completes the proof of Theorem 3.

Remark 1: If p(t,x,y,z) = e(t) then (1) reduces to the case studied by Swick[17]. Clearly our results improve and extend that of [17].

Remark 2: Unlike in[17] and[21], the bounding constants in Theorem 2 and Theorem 3 do not depend on the solutions of (1) and (5).

CONCLUSION

It is well known that the problem of ultimate boundedness of solutions of nonlinear is very important in the theory and applications of differential equations. And the effective method for studying problems of ultimate boundedness of solution of nonlinear differential equations is still the Lyapunov's direct method see for instance[1,7-15,18-23]. In this study a complete Lyapunov function was used to achieve the desired results.

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