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STABILITY AND UNIFORM ULTIMATE BOUNDEDNESS OF SOLUTIONS OF A THIRD-ORDER DIFFERENTIAL EQUATION

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Abstract: This paper is concerned with uniform ultimate boundedness of solutions of a third-order nonlinear differential equation (1.1). Sufficient conditions under which all solutions $x(t)$, its first and second derivatives tend to zero as $t \to \infty$, when $p(t, x, x', x'') \equiv 0$, are presented.

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1. Introduction

We shall be concerned here, with stability of the zero and ultimate boundedness of solutions of a third-order nonlinear differential equation

$$x''' + \psi(t)f(x, x', x'')x'' + \phi(t)g(x, x') + \varphi(t)h(x, x', x'') = p(t, x, x', x''), \quad (1.1)$$

or its equivalent system

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$$\begin{aligned} x' &= y, y' = z, \\ z' &= p(t,x,y,z) - \psi(t)f(x,y,z)z - \phi(t)g(x,y) + \varphi(t)h(x,y,z) \end{aligned} \quad (1.2)$$

in which $p \in C(\mathbb{R}^+ \times \mathbb{R}^3, \mathbb{R})$; $f, h \in C(\mathbb{R}^3, \mathbb{R})$; $g \in C(\mathbb{R}^2, \mathbb{R})$; $\psi, \phi, \varphi \in C(\mathbb{R}^+, \mathbb{R})$; $\mathbb{R} = (-\infty, \infty)$; $\mathbb{R}^+ = [0, \infty)$; $\psi, \phi, \varphi, f, g, h$ and p depend only on the arguments displaced explicitly and $\frac{\partial}{\partial x} f(x,y,z) = f_x(x,y,z)$, $\frac{\partial}{\partial y} f(x,y,z) = f_y(x,y,z)$, $\frac{\partial}{\partial z} f(x,y,z) = f_z(x,y,z)$, $\frac{\partial}{\partial x} g(x,y) = g_x(x,y)$, $\frac{\partial}{\partial x} h(x,y,z) = h_x(x,y,z)$, $\frac{\partial}{\partial y} h(x,y,z) = h_y(x,y,z)$, $\frac{\partial}{\partial z} h(x,y,z) = h_z(x,y,z)$, $\frac{d}{dt}\psi(t) = \psi'(t)$, $\frac{d}{dt}\phi(t) = \phi'(t)$ and $\frac{d}{dt}\varphi(t) = \varphi'(t)$ exist and are continuous for all x, y, z and t. As usual, condition for uniqueness will be assumed and x', x'', x'''as elsewhere, stand for differentiation with respect to the independent variable t.

Equation (1.2), for $p(t,x,y,z) = 0$, $p(t,x,y,z) = p(t)$ and $p(t,x,y,z) \neq 0$, have been the object of a good deal of research over the past several years. See for instance Reissig *et. al.* [8], Ademola, *et. al.* [1, 2], Afuwape [3], Bereketoğlu and Györi [4], Ezeilo [5], Ezeilo and Tejumola [6], Omeike [7], Swick [9], Tunç [10] and the references therein. These works were done with the aid of Lyapunov functions or Yoshizawa functions except in [3], where frequency domain approach was used.

In [10] Tunç established conditions for boundedness of solutions of a third-order nonlinear third-order nonlinear differential equation

$$x''' + f(x,x',x'')x'' + g(x,x') + h(x,x',x'') = p(t,x,x',x''). \quad (1.3)$$

Recently, Ademola, *et. al.* [1] and Omeike [7] studied conditions under which all solutions of the third-order differential equation (1.3) were ultimately bounded using a complete Yoshizawa and a complete Lyapunov functions respectively. However, the problem of stability and ultimate boundedness of solutions in which the nonlinear terms (the restoring terms in particular) are multiple of functions of t, are scarce.

Our aim in this paper is to study uniform boundedness and conditions under which all solutions $x(t)$, its first and second derivatives tend to zero as $t \to \infty$ when $p(t,x,x',x'') = 0$ in (1.1). We also established conditions for uniform ultimate boundedness of solutions of equation (1.1). Our results generalize many results which have been discussed in [8] and include the result in [7]. This work is motivated from the works of Ademola, *et. al.* [2], Omeike [7] and Tunç [10].

2. Main Results

In the case $p(t,x,y,z) \equiv 0$, equation (1.2) becomes

$$x' = y, \quad y' = z, \quad z' = -\psi(t)f(x,y,z)z - \phi(t)g(x,y) - \varphi(t)h(x,y,z) \qquad (2.1)$$

with the following result.

Theorem 1. *Further to the basic assumptions on the functions f, g, h, ψ, ϕ and φ, suppose that $a, a_1, b, b_1, c, \delta_0, \epsilon_0, \psi_0, \psi_1, \phi_0, \phi_1, \varphi_0$ and φ_1 are positive constants and that:*

(i) $\psi_0 \le \psi(t) \le \psi_1$, $\phi_0 \le \phi(t) \le \phi_1$ and $\varphi_0 \le \varphi(t) \le \varphi_1$ for all $t \ge 0$;

(ii) $h(0,0,0) = 0$, $\delta_0 \le h(x,y,z)/x$ for all $x \ne 0, y$ and z;

(iii) $h_x(x,0,0) \le c$ for all x;

(iv) $g(0,0) = 0$, $b \le g(x,y)/y \le b_1$ for all $x, y \ne 0$;

(v) $a \le f(x,y,z) \le a_1$ for all x, y, z and $ab > c$;

(vi) $\sup_{t\ge0}[|\psi'(t)| + |\phi'(t)| + |\varphi'(t)|] < \epsilon_0$;

(vii) $g_x(x,y) \le 0$, $yf_x(x,y,z) \le 0$ for all x, y;

(viii) $h_y(x,y,0) \ge 0$, $h_z(x,0,z) \ge 0$, $yf_z(x,y,z) \ge 0$ for all x, y, z.

Then every solution $(x(t), y(t), z(t))$ of (2.1) is uniform-bounded and satisfies $x(t) \to 0$, $y(t) \to 0$, $z(t) \to 0$ as $t \to \infty$.

Remark 2. The hypotheses: $\psi(t) \ge \psi_0$, $\phi(t) \ge \phi_0$, $\varphi(t) \le \varphi_1$, $h(x,0,0)/x \ge \delta_0$ $x \ne 0$, $g(x,y)/y \ge b$ $y \ne 0$, $h_x(x,0,0) \le c$ and $f(x,y,z) \ge a$ imply the existence of positive constants α and β, satisfying

$$\frac{\varphi_1 c}{\phi_0 b} < \alpha < \psi_0 a \qquad (2.2a)$$

and

$$\beta < \min\left\{(ab\psi_0\phi_0 - c\varphi_1)\eta_1; b\phi_0; \frac{1}{2}(a\psi_0 - \alpha)\eta_2\right\}, \qquad (2.2b)$$

where

$$\eta_1 = \left[1 + a\psi_1 + \delta_0^{-1}\varphi_0^{-1}\phi_0^2[\frac{g(x,y)}{y} - b]^2\right]^{-1}$$

and

$$\eta_2 = \left[1 + \delta_0^{-1}\varphi_0^{-1}\psi_0^2[f(x,y,z) - a]^2\right]^{-1}$$

are generalization of Routh-Hurwitz stability criteria.

Remark 3. (i) If $\psi(t) \equiv \phi(t) \equiv \varphi(t) \equiv 1$, $f(x,y,z)z \equiv f(z)$, $g(x,y) \equiv g(y)$ and $h(x,y,z) \equiv h(x)$, then the conclusion of Theorem 1 coincides with those of Ademola in [2].

(ii) Whenever $\psi(t)f(x,y,z) \equiv f(t,x,y)$, $\phi(t)g(x,y) \equiv r(t)g(y)$, and $\psi(t)h(x,y,z) \equiv q(t)h(x)$ also, the conclusion of Theorem 1 coincides with that of Swick in [9].

(iii) Moreover, hypotheses of Theorem 1 (in particular on functions h and f) are less restrictive than those in [2] and [9], respectively.

In what follows, $D, D_0, D_1, \cdots, D_{15}$ denote finite positive constants whose magnitudes depend only on $a, a_1, b, b_1, c, \delta_0, \delta_1, \psi_0, \psi_1, \phi_0, \phi_1, \varphi_0, \varphi_1, \epsilon_0, \epsilon_1, P_0, P_1$ and ρ. The D's without suffixes are not necessarily the same each time they occur, but each of the numbered D's: $D_0, D_1, \cdots, D_{15}$ retains a fixed identity throughout.

The proofs of the above and the subsequent results depend on a continuously differentiable function $V = V(t,x,y,z)$ defined by

$$\begin{aligned} 2V &= 2[\alpha + a\psi(t)]\varphi(t)\int_0^x h(\xi,0,0)d\xi + 4y\varphi(t)h(x,0,0) + 2a\beta\psi(t)xy \\ &+ 4\phi(t)\int_0^y g(x,\tau)d\tau + 2[\alpha + a\psi(t)]\psi(t)\int_0^y \tau f(x,\tau,0)d\tau + 2z^2 \\ &+ \beta y^2 + b\beta\phi(t)x^2 + 2\beta xz + 2[\alpha + a\psi(t)]yz, \end{aligned} \tag{2.3}$$

where α and β are positive constants defined in (2.2a) and (2.2b) respectively. This function and its derivative with respect to the independent variable t, satisfies some fundamental inequalities as seen in the following lemmas.

Lemma 4. *Subject to assumptions (i)-(v) of Theorem 1, $V(t,0,0,0) = 0$ and there exist positive constants $D_0 = D_0(a,b,c,\alpha,\beta,\delta_0,\psi_0,\phi_0,\varphi_0,\varphi_1)$ and $D_1 = D_1(a,b,c,a_1,b_1,\alpha,\beta,\psi_1,\phi_1,\varphi_1)$ such that*

$$D_0(x^2(t) + y^2(t) + z^2(t)) \leq V(t,x,y,z) \leq D_1(x^2(t) + y^2(t) + z^2(t))$$

and

$$V(t,x,y,z) \to \infty \text{ as } x^2(t)+y^2(t)+z^2(t) \to \infty.$$

Proof. Clearly $V(t,0,0,0)=0$. Since $b \neq 0 \neq \phi(t)$ and $h(0,0,0)=0$, by hypotheses of Theorem 1, then equation (2.3) can be recast in the form

$$\begin{aligned} V = &\int_0^y \Big\{[\alpha + a\psi(t)]\psi(t)f(x,\tau,0) - [\alpha^2 + a^2\psi^2(t)]\Big\}\tau d\tau + \frac{1}{2}(\alpha y + z)^2 \\ &+ \frac{\varphi(t)}{b\phi(t)}\int_0^x \Big\{[\alpha + a\psi(t)]b\phi(t) - 2\varphi(t)h_\xi(\xi,0,0)\Big\}h(\xi,0,0)d\xi \\ &+ \frac{1}{2}\beta y^2 + 2\phi(t)\int_0^y \left[\frac{g(x,\tau)}{\tau} - b\right]\tau d\tau + \frac{1}{2}\Big(\beta x + a\psi(t)y + z\Big)^2 \\ &+ \frac{1}{2}\beta[b\phi(t)-\beta]x^2. \end{aligned} \tag{2.4}$$

In view of hypotheses (i) and (v) of Theorem 1, $\psi(t) \geq \psi_0$, $\phi(t) \geq \phi_0$ and $f(x,y,0) \geq a$ for all x, y and $t \geq 0$, so that

$$\int_0^y \{[\alpha + a\psi(t)]\psi(t)f(x,\tau,0) - [\alpha^2 + a^2\psi^2(t)]\}\tau d\tau \geq \frac{1}{2}\alpha(a\psi_0 - \alpha)y^2. \tag{2.5a}$$

From hypotheses (i)-(iii) of Theorem 1, $\psi(t) \geq \psi_0$, $\phi(t) \geq \phi_0$, $\varphi(t) \leq \varphi_1$ $h(x,0,0)/x \geq \delta_0$ and $h_x(x,0,0) \leq c$ so that

$$\int_0^x \{[\alpha + a\psi(t)]b\phi(t) - 2\varphi(t)h_\xi(\xi,0,0)\}h(\xi,0,0)d\xi \geq \eta_3 x^2, \tag{2.5b}$$

where $\eta_3 = \frac{1}{2}\{(\alpha + a\psi_0)b\phi_0 - 2c\varphi_1\}\delta_0$. Finally, since $\phi(t) \geq \phi_0$, we obtain

$$(b\phi(t) - \beta)x^2 \geq (b\phi_0 - \beta)x^2. \tag{2.5c}$$

On gathering estimates (2.5a)-(2.5c), into (2.4), we obtain

$$\begin{aligned} V \geq &\frac{1}{2}\{b^{-1}\phi_0^{-1}\delta_0\varphi_0[(\alpha + a\psi_0)b\phi_0 - 2c\varphi_1] + \beta(b\phi_0 - \beta)\}x^2 + \frac{1}{2}(\alpha y + z)^2 \\ &+ \frac{1}{2}[\alpha(a\psi_0 - \alpha) + \beta]y^2 + b^{-1}\phi_0^{-1}[b\phi_0 y + \varphi_0\delta_0 x]^2 + \frac{1}{2}[\beta x + a\psi_0 y + z]^2. \end{aligned} \tag{2.6}$$

In view of (2.2a) and (2.2b), we have $a\psi_0 > \alpha$, $ab\psi_0\phi_0 > c\varphi_1$ and $b\phi_0 > \beta$, such that estimate (2.6) is positive definite, thus there exists a positive constant D_2 such that

$$V \geq D_2(x^2 + y^2 + z^2). \tag{2.7}$$

Now, to establish the upper inequality of Lemma 4, condition (iii) of Theorem 1 implies that $h(x,0,0) \leq cx$ for all $x \neq 0$ since $h(0,0,0) = 0$. Also, in view of conditions (i), (iv), (v) of Theorem 1 and Schwartz inequality, equation (2.3) becomes

$$V \leq \eta_4 x^2 + \eta_5 y^2 + \eta_6 z^2,$$

where $\eta_4 = \frac{1}{2}[(a\psi_1 + b\phi_1 + 1)\beta + (\alpha + a\psi_1 + 2)c\varphi_1]$, $\eta_5 = \frac{1}{2}[[(\alpha + a\psi_1)a_1 + a(\beta + 1)]\psi_1 + 2b_1\phi_1 + 2c\varphi_1 + \alpha + \beta]$ and $\eta_6 = \frac{1}{2}[a\psi_1 + \alpha + \beta + 2]$. Hence, there is a positive constant $D_3 = \max\{\eta_4, \eta_5, \eta_6\}$ such that

$$V \leq D_3(x^2 + y^2 + z^2).$$

From estimate (2.7), it follows that $V(t,0,0,0) = 0$ if and only if $x^2+y^2+z^2 = 0$ and $V(t,x,y,z) > 0$ for $x^2 + y^2 + z^2 \neq 0$ and hence

$$V(t,x,y,z) \to \infty \text{ as } x^2 + y^2 + z^2 \to \infty.$$

This completes the proof of Lemma 4. □

Lemma 5. *Under the hypotheses of Theorem 1, there is a positive constant $D = D(a,b,c,\delta_0,\epsilon,\psi_0,\phi_0,\varphi_0,\varphi_1,\alpha,\beta)$ such that along a solution of* (2.1)

$$V' = \frac{d}{dt}V(t,x,y,z) \leq -D(x^2(t) + y^2(t) + z^2(t)) \leq 0.$$

Proof. Along any solution $(x(t), y(t), z(t))$ of (2.1), we have

$$\begin{aligned} V'_{(2.1)} &= W_1 + W_2 + W_3 - (W_4 + W_5) - \beta\phi(t)\left[\frac{g(x,y)}{y} - b\right]xy \\ &\quad - \beta\psi(t)[f(x,y,z) - a]xz, \end{aligned} \tag{2.8}$$

where

$$W_1 := a\beta\psi(t)y^2 + 2\beta yz;$$

$$W_2 := 2\phi(t)y\int_0^y g_x(x,\tau)d\tau + [\alpha + a\psi(t)]\psi(t)y\int_0^y \tau f_x(x,\tau,0)d\tau;$$

$$\begin{aligned} W_3 := &\{[\alpha + a\psi(t)]\varphi'(t) + a\psi'(t)\varphi(t)\}\int_0^x h(\xi,0,0)d\xi + 2\phi'(t)\int_0^y g(x,\tau)d\tau \\ &+ 2\varphi'(t)h(x,0,0)y + [\alpha + 2a\psi(t)]\psi'(t)\int_0^y \tau f(x,\tau,0)d\tau \\ &+ \frac{1}{2}b\beta\phi'(t)x^2 + a\beta\psi'(t)xy + a\psi'(t)yz; \end{aligned}$$

$$W_4 := [\alpha+a\psi(t)]\varphi(t)y^2\left[\frac{h(x,y,z)-h(x,0,0)}{y}\right]+2\varphi(t)z^2\left[\frac{h(x,y,z)-h(x,0,0)}{z}\right]$$
$$+[\alpha+a\psi(t)]\psi(t)yz^2\left[\frac{f(x,y,z)-f(x,y,0)}{z}\right]$$

and

$$W_5 := \beta\varphi(t)\frac{h(x,y,z)}{x}x^2+\left[[\alpha+a\psi(t)]\phi(t)\frac{g(x,y)}{y}-2\varphi(t)h_x(x,0,0)\right]y^2$$
$$+\left[2\psi(t)f(x,y,z)-[\alpha+a\psi(t)]\right]z^2.$$

Now, from the obvious inequality $2|p||q| \le p^2+q^2$ and $\psi(t)\le\psi_1$, we have

$$W_1 \le \beta[(a\psi_1+1)y^2+z^2].$$

By hypothesis (vii) of Theorem 1, we obtain

$$W_2 \le 0.$$

Furthermore, $h(0,0,0)=0$ implies that $h(x,0,0)/x \le c$ for $x\neq 0$. Also $\psi(t)\le \psi_1$, $\phi(t)\le\phi_1$, $\varphi(t)\le\varphi_1$, $g(x,y)/y\le b_1$ for all $y\neq 0$ and $f(x,y,0)\le a_1$. With these conditions, we have

$$W_3 \le \left[\frac{1}{2}ac(\varphi_1+\beta)|\psi'(t)|+\frac{1}{2}b\beta|\phi'(t)|+\frac{1}{2}c[\alpha+a\psi_1+2]|\varphi'(t)|\right]x^2$$
$$+\left[\frac{1}{2}[a_1(\alpha+2a\psi_1)+a(\beta+1)]|\psi'(t)|+b_1|\phi'(t)|+c|\varphi'(t)|\right]y^2+\frac{1}{2}a|\psi'(t)|z^2.$$

Thus, there are positive constants D_4, D_5, D_6 such that

$$W_3 \le \max\{D_4,\ D_5,\ D_6\}[|\psi'(t)|+|\phi'(t)|+|\varphi'(t)|](x^2+y^2+z^2),$$

where $D_4=\frac{1}{2}\max\{ac(\varphi_1+\beta),\ b\beta,\ c(\alpha+a\psi_1)\}$, $D_5=\max\{\frac{1}{2}[a_1(\alpha+2a\psi_1)+a(\beta+1)],\ b_1,\ c\}$ and $D_6=\frac{1}{2}a$.

By assumption (viii) of Theorem 1 for $y\neq 0$, we have

$$[\alpha+a\psi(t)]\varphi(t)y^2\left[\frac{h(x,y,z)-h(x,0,0)}{y}\right]$$
$$=[\alpha+a\psi(t)]\varphi(t)y^2h_y(x,\theta_1 y,0)\ge 0, \qquad (2.9a)$$

$0\le\theta_1\le 1$ and when $y=0$, $[\alpha+a\psi(t)]\varphi(t)y^2h_y(x,\theta_1 y,0)=0.$

Similarly, for $z \neq 0$, we have

$$2\varphi(t)z^2\left[\frac{h(x,y,z)-h(x,0,0)}{z}\right] = 2\varphi(t)z^2 h_z(x,0,\theta_2 z) \geq 0, \tag{2.9b}$$

$0 \leq \theta_2 \leq 1$ and $2\varphi(t)z^2 h_z(x,0,\theta_2 z) = 0$ when $z = 0$. Also for $z \neq 0$, we obtain

$$\begin{aligned}&[\alpha + a\psi(t)]\psi(t)yz^2\left[\frac{f(x,y,z)-f(x,y,0)}{z}\right]\\ &\qquad = [\alpha + a\psi(t)]\psi(t)yz^2 f_z(x,y,\theta_3 z) \geq 0,\end{aligned} \tag{2.9c}$$

$0 \leq \theta_3 \leq 1$ and $[\alpha + a\psi(t)]\psi(t)yz^2 f_z(x,y,\theta_3 z) = 0$ when $z = 0$.
A combination of (2.9a), (2.9b) and (2.9c) yields

$$W_4 \geq 0.$$

Also, by hypotheses (i) and (ii) of Theorem 1, we obtain

$$\beta\varphi(t)h(x,y,z)x \geq \beta\delta_0\varphi_0 x^2. \tag{2.10a}$$

Since $\psi(t) \geq \psi_0$, $\phi(t) \geq \phi_0$, $\varphi(t) \leq \varphi_1$, $h_x(x,0,0) \leq c$ and $g(x,y)/y \geq b$ for all $x, y \neq 0$, we have

$$[\alpha + a\psi(t)]\phi(t)\frac{g(x,y)}{y} - 2\varphi(t)h_x(x,0,0) \geq (\alpha + a\psi_0)b\phi_0 - 2c\varphi_1. \tag{2.10b}$$

By conditions (i) and (v) of Theorem 1, we find that

$$2\psi(t)f(x,y,z) - [\alpha + a\psi(t)] \geq a\psi_0 - \alpha. \tag{2.10c}$$

Combining estimates (2.10a), (2.10b) and (2.10c), we have

$$W_5 \geq \beta\delta_0\varphi_0 x^2 + [(\alpha + a\psi_0)b\phi_0 - 2c\varphi_1]y^2 + (a\psi_0 - \alpha)z^2.$$

On gathering estimates W_i $(i = 1,2,3,4,5)$ with (2.8), we obtain

$$\begin{aligned}V'_{(2.1)} \leq &-\frac{1}{2}\beta\delta_0\varphi_0 x^2 - [(\alpha + a\psi_0)b\phi_0 - 2c\varphi_1 - \beta(a\psi_1 + 1)]y^2\\ &- (a\psi_0 - \alpha - \beta)z^2 - (W_6 + W_7)\\ &+ D_7[|\psi'(t)| + |\phi'(t)| + |\varphi'(t)|](x^2 + y^2 + z^2),\end{aligned} \tag{2.11}$$

where $W_6 = \frac{1}{4}\beta\delta_0\varphi_0 x^2 + \beta\phi_0\left[\frac{g(x,y)}{y} - b\right]xy$, $W_7 = \frac{1}{4}\beta\delta_0\varphi_0 x^2 + \beta\psi_0[f(x,y,z) - a]xz$ and $D_7 = \max\{D_4, D_5, D_6\}$. On completing the squares, we have

$$W_6 =\geq -\beta\delta_0^{-1}\varphi_0^{-1}\phi_0^2\left[\frac{g(x,y)}{y} - b\right]^2 y^2 \tag{2.12a}$$

and
$$W_7 \geq -\beta\delta_0^{-1}\varphi_0^{-1}\psi_0^2[f(x,y,z)-a]^2z^2, \tag{2.12b}$$
since β, δ_0, φ are positive constants, it follows that $[x+2\delta_0^{-1}\varphi_0^{-1}\phi_0[\frac{g(x,y)}{y}-b]y]^2 \geq 0$ and $[x+2\delta_0^{-1}\varphi_0^{-1}\psi_0[f(x,y,z)-a]z]^2 \geq 0$ for all x, y, z. Substituting (2.12a) and (2.12b) into (2.11) and by (2.2b), we obtain
$$V'_{(3.2)} \leq -D_8(x^2+y^2+z^2) + D_7[|\psi'(t)| + |\phi'(t)| + |\varphi'(t)|](x^2+y^2+z^2), \tag{2.13}$$
where $D_8 = \min\{\frac{1}{2}\beta\delta_0\varphi_0,\ \alpha b\phi_0 - c\varphi_1,\ \frac{1}{2}(a\psi_0-\alpha)\}$.

Finally, by condition (vi) of Theorem 1, choose ϵ_0 sufficiently small such that $D_8 > D_7\epsilon_0$, then we can find a positive constant D_9 such that
$$V'_{(2.1)} \leq -D_9(x^2+y^2+z^2) \leq 0$$
for all x, y and z. This completes the proof of Lemma 5. □

Proof of Theorem 1. Let $(x(t), y(t), z(t))$ be any solution of (2.1). From Lemma 4 and Lemma 5 all solutions of (2.1) are uniform bounded (see p. 38-39 in [11]). Furthermore, from Lemma 5, we have $V' \leq -D_9(x^2+y^2+z^2)$. Let $W(X) \equiv D_9(x^2+y^2+z^2)$, a positive definite function with respect to a closed set $\Omega \equiv \{(x,y,z)|x=0, y=0, z=0\}$, then $V' \leq -W(X)$. Since $h(x,y,z)$, is continuous for all x, y, z and functions $\psi(t), \phi(t), \varphi(t), f(x,y,z)$ and $g(x,y)$ are bounded above, it follows that
$$\|F(t,X)\| = \left\| \begin{pmatrix} y \\ z \\ -\varphi(t)h(x,y,z) - \phi(t)g(x,y) - \psi(t)f(x,y,z)z \end{pmatrix} \right\|$$
is bounded for all t when X belongs to any compact subset of $\mathbb{R}^3$. Since $x = 0, y = 0, z = 0$ on the set Ω, it follows from Theorem 14.1 p.60-61 in [11] that $x(t) \to 0$, $y(t) \to 0$, $z(t) \to 0$ as $t \to \infty$. □

Theorem 6. *Suppose that* $a, b, c, \delta_0, \epsilon_0, \epsilon_1, \psi_0, \psi_1, \phi_0, \varphi_0, \varphi_1$ *are positive constants and* $P_1 \geq 0$ *so that:*

(i) *hypotheses* (i)-(viii) *of Theorem 1 hold;*

(ii) $|p(t,x,y,z)| \leq p_1(t) + p_2(t)(|x|+|y|+|z|)$ *where* $p_1(t)$ *and* $p_2(t)$ *are non-negative continuous functions satisfying*
$$0 \leq p_1(t) \leq P_1 \tag{2.14}$$
and
$$0 \leq p_2(t) \leq \epsilon_1. \tag{2.15}$$

Then the solution $(x(t), y(t), z(t))$ of (1.2) is uniformly ultimately bounded.

Lemma 7. *Subject to the conditions of Theorem 2.6 there exists positive constant D_{10} depending only on a, b, c, δ_0, $\psi_0, \psi_1, \phi_0, \varphi_0, \varphi_1$, $\epsilon_0, \epsilon_1, \alpha$, β and P_1 such that for any solution $(x(t), y(t), z(t))$ of (1.2)*

$$V' \equiv \frac{d}{dt}V(t, x(t), y(t), z(t)) \leq -D_{10}(x^2 + y^2 + z^2).$$

Proof. Along a solution $(x(t), y(t), z(t))$ of (1.2), we have

$$V'_{(1.2)} = V'_{(2.1)} + [\beta x + [\alpha + a\psi(t)]y + 2z]p(t, x, y, z).$$

In view of (2.13), hypotheses (vi) of Theorem 1 and (ii) of Theorem 6, we find

$$V'_{(1.2)} \leq -D_8(x^2 + y^2 + z^2) + D_{11}(|x| + |y| + |z|)|p(t, x, y, z)|$$
$$+D_7(|\psi'(t)| + |\phi'(t)| + |\varphi'(t)|)(x^2 + y^2 + z^2)$$
$$\leq -D_8(x^2 + y^2 + z^2) + D_7\epsilon_0(x^2 + y^2 + z^2)$$
$$+D_{11}(|x| + |y| + |z|)[p_1(t) + p_2(t)(|x| + |y| + |z|)],$$

where $D_{11} = \max\{\beta, \alpha + a\psi_0, 2\}$. By (2.14) and (2.15) and the Schwartz inequality, we obtain

$$V'_{(1.2)} \leq -(D_8 - D_7\epsilon_0 - 3D_{11}\epsilon_1)(x^2 + y^2 + z^2) + 3^{1/2}P_1D_{11}(x^2 + y^2 + z^2)^{1/2}.$$

Again choose ϵ_0 and ϵ_1 so small so that $D_8 > D_7\epsilon_0 + 3D_{11}\epsilon_1$ then there exist positive constants D_{12} and D_{13} such that

$$V'_{(1.2)} \leq -D_{12}(x^2 + y^2 + z^2) + D_{13}(x^2 + y^2 + z^2)^{1/2}. \tag{2.16}$$

Choose $(x^2 + y^2 + z^2)^{1/2} \geq 2D_{12}^{-1}D_{13} = D_{14}$ the inequality in (2.16) becomes

$$V'_{(1.2)} \leq -D_{15}(x^2 + y^2 + z^2),$$

where $D_{15} = \frac{1}{2}D_{12}$. □

Proof. of Theorem 2.6. The proof of Theorem 2.6 follows from Lemma 4, Lemma 7 and Theorem 10.4, p. 42 in [11] that the solution $(x(t), y(t), z(t))$ of (1.2) is uniform ultimately bounded. □

Remark 8. As usually, if $\psi(t)f(x,x',x'') = a$, $\phi(t)g(x,x') = bx'$, $\varphi(t)h(x,x',x'') = cx$ and $p(t,x,x',x'') = 0$ in (1.1) all hypotheses of Theorem 1 reduce to

$$a > 0, b > 0, c > 0, ab - c > 0$$

which is the Routh-Hurwitz criterion for the global asymptotic stability of the zero solution of the equation

$$x''' + ax'' + bx' + cx = 0.$$

Remark 9. If $\psi(t) \equiv \phi(t) \equiv \varphi(t) \equiv 1$ and $p_2(t) = 0$, then system (1.2) reduces to that studied by Omeike in [7], thus our result includes that of [7]. In addition, the hypothesis on the function $f(x,y,z)$ is weaker than those used by Omeike in [7], since there it was required that $f(x,y,z) > 0$. Hence, our result generalizes that of [7].

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