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Stability, Boundedness and periodic solutions to certain second order delay differential equations

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Abstract

Stability, boundedness and existence of a unique periodic solution to certain second order nonlinear delay differential equations is discussed. By employing Lyapunov's direct (or second) method, a complete Lyapunov functional is constructed and used to establish sufficient conditions, on the nonlinear terms, that guarantee uniform asymptotic stability, uniform ultimate boundedness and existence of a unique periodic solution. Obtained results complement many outstanding recent results in the literature. Finally, examples are given to show the effectiveness of our method and correctness of our results.

Subjclass [2000] : *34C25, 34D20, 34D40, 65L06.*

Keywords : *Second order; Nonlinear differential equation; Uniform stability; Uniform ultimate boundedness; Existence of a unique periodic solutions*

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1. Introduction

Qualitative behaviour of solutions of various second order differential equations with and without delay have been extensively discussed in the literature and are still receiving attention of authors because of their practical applications. Readers are referred to the books of Burton et al. [5,6], Driver [8], Hale [11], Kolmanovskii and Myshkis [13], Kuag [15], Lakshmikantham et. al. [16], Yoshizawa [30,31,32], which contain general results on the subject matter and the papers of Ademola [1], Ademola et. al. [2,3], Alaba and Ogundare [4], Domoshnitsky [9], Grioryan [10], Jin and Zengrong [12], Kroopnick [14], Ogundare et. al. [17], Ogundare and Afuwape [18], Ogundare and Okecha [19], Tunç [20]-[24], Wang and Zhu [26], Xu [27], Yeniçerioğlu [28,29] and the references cited therein.

In [18] the authors discussed boundedness and stability properties of solutions of

$$x'' + f(x)x' + g(x) = p(t, x, x'),$$

where f, g and p are continuous functions in their respective arguments t, x and x'. In [20] the author discussed boundedness of solutions to

$$x'' + c(t, x, x') + q(t)b(x) = f(t)$$

where c, b, q and f are continuous functions defined on $\mathbf{R}^+ \times \mathbf{R}^2$, $\mathbf{R}$, $\mathbf{R}^+$ and $\mathbf{R}^+$ respectively. Recently, in [4] the authors studied the second order non autonomous damped and forced nonlinear ordinary differential equation of the form

$$[a(t)x']' + b(t)f(x, x')x' + c(t)g(x) = p(t, x, x'),$$

where the functions a, b, c, f, g and p depend only on the arguments displayed explicitly.

Finally in [1] the author considered stability, boundedness and existence of unique periodic solutions to the following second order ordinary differential equation

$$[\phi(x)x']' + g(t, x, x')x' + \varphi(t)h(x) = p(t, x, x')$$

where ϕ, g, φ, h and p are continuous functions in their respective arguments on $\mathbf{R}, \mathbf{R}^+ \times \mathbf{R}^2, \mathbf{R}^+, \mathbf{R}$ and $\mathbf{R}^+ \times \mathbf{R}^2$ respectively.

Unfortunately, the problem of uniform asymptotic stability (when the function $p = 0$), uniform boundedness, uniform ultimate boundedness, existence and uniqueness of periodic solutions to second order nonlinear delay

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differential equation (1.1), where the nonlinear functions g, h and p contain variable deviating arguments and the second ordered derivative contains a variable coefficient, is yet to be investigated. The purpose of this paper therefore is to fill this gap. We will consider

$$[\phi(x(t))x'(t)]' + g(t, x(t-\tau(t)), x'(t-\tau(t)))x'(t) + h(x(t-\tau(t)))$$

$$= p(t, x(t-\tau(t)), x'(t-\tau(t))), \quad (1.1)$$

where ϕ, g, h, p and τ are continuous functions in their respective arguments, (i.e. $\phi, h : \mathbf{R} \to \mathbf{R}$, $\tau : \mathbf{R}^+ \to \mathbf{R}^+$ and $g, p : \mathbf{R}^+ \times \mathbf{R}^2 \to \mathbf{R}$ with $\mathbf{R} := (-\infty, \infty)$ and $\mathbf{R}^+ := [0, \infty)$). The primes indicate differentiation with respect to the independent variable t. If $x'(t) = y(t)\phi^{-1}(x(t))$, $\phi(x(t)) \neq 0$ for all $t \geq 0$, then equation (1.1) is equivalent to system of first order delay differential equations

$$x'(t) = y(t)\phi^{-1}(x(t)),$$

$$y'(t) = p(t, x(t-\tau(t)), y(t-\tau(t))\phi^{-1}(x(t-\tau(t))))$$

$$+ \int_{t-\tau(t)}^{t} h'(x(s))y(s)\phi^{-1}(x(s))ds - h(x)$$

$$- g(t, x(t-\tau(t)), y(t-\tau(t))\phi^{-1}(x(t-\tau(t))))y(t)\phi^{-1}(x(t)), \quad (1.2)$$

where $0 \leq \tau(t) \leq \alpha$, $\alpha > 0$ is a constant to be determined later and the derivatives h', ϕ' and τ' exist and continuous for all x and t. The work is motivated by the recent works in [1,3,4,20]. The obtained results are new, in fact according to our observation from relevant literature, this is the first paper on second order delay differential equations where the highest ordered derivatives contains variable coefficient. In Section 2 we discussed the basic mathematical tools that will be used in the sequel. In Section 3 the main results are stated and proved while examples are given in Section 4.

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2. Preliminary Results

Consider the following general nonlinear non-autonomous delay differential equation

$$(2.1)\qquad \dot{X} = \frac{dX}{dt} = F(t, X_t), \quad X_t = X(t+\theta), \quad -r \leq \theta < 0, \quad t \geq 0$$

where $F : \mathbf{R}^+ \times C_H \to \mathbf{R}^n$ is a continuous mapping, $F(t+\omega, \phi) = F(t, \phi)$ for all $\phi \in C$ and for some positive constant ω. We assume that F takes closed bounded sets into bounded sets in $\mathbf{R}^n$. $(C, \|\cdot\|)$ is the Banach space of continuous function $\varphi : [-r, 0] \to \mathbf{R}^n$ with supremum norm, $r > 0$; for $H > 0$, we define $C_H \subset C$ by $C_H = \{\varphi \in C : \|\varphi\| < H, \}$ C_H is the open $H-$ball in C, $C = C([-r, 0], \mathbf{R}^n)$. We shall state the following basic results:

Lemma 2.1. *(See [32] pp 206). Suppose that $F(t, \phi) \in \overline{C}_0(\phi)$ and $F(t, \phi)$ is periodic in t of period ω, $\omega \geq r$, and consequently for any $\alpha > 0$ there exists an $L(\alpha) > 0$ such that $\phi \in C_\alpha$ implies $|F(t, \phi)| \leq L(\alpha)$. Suppose that a continuous Lyapunov functional $V(t, \phi)$ exists, defined on $t \in \mathbf{R}^+$, $\phi \in S^*$, S^* is the set of $\phi \in C$ such that with $|\phi(0)| \geq H$ (H may be large) and that $V(t, \phi)$ satisfies the following conditions:*

(i) $a(|\phi(0)|) \leq V(t, \phi) \leq b(\|\phi\|)$, where $a(r)$ and $b(r)$ are continuous, increasing and positive for $r \geq H$ and $a(r) \to \infty$ as $r \to \infty$;

(ii) $\dot{V}_{(2.1)}(t, \phi) \leq -c(|\phi(0)|)$, where $c(r)$ is continuous and positive for $r \geq H$.

Suppose that there exists an $H_1 > 0$, $H_1 > H$, such that

$$(2.2)\qquad hL(\gamma^*) < H_1 - H,$$

where $\gamma^* > 0$ is a constant which is determined in the following way: By the condition on $V(t, \phi)$ there exist $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ such that $b(H_1) \leq a(\alpha)$, $b(\alpha) \leq a(\beta)$ and $b(\beta) \leq a(\gamma)$. γ^* is defined by $b(\gamma) \leq a(\gamma^*)$. Under the above conditions, there exists a periodic solution of (2.1) of period ω. In particular, the relation (2.2) can always be satisfied if h is sufficiently small.

Lemma 2.2. *(See [32] pp 188).* Suppose that $F(t, \phi)$ is defined and continuous on $0 \leq t \leq c$, $\phi \in C_H$ and that there exists a continuous Lyapunov functional $V(t, \phi, \varphi)$ defined on $0 \leq t \leq c$, $\phi, \varphi \in C_H$ which satisfy the following conditions:

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(i) $V(t, \phi, \varphi) = 0$ if $\phi = \varphi$;

(ii) $V(t, \phi, \varphi) > 0$ if $\phi \neq \varphi$;

(iii) for the associated system

$$\dot{x}(t) = F(t, x_t), \qquad \dot{y}(t) = F(t, y_t) \tag{2.3}$$

we have $V'_{(2.3)}(t, \phi, \varphi) \leq 0$, where for $\|\phi\| = H$ or $\|\varphi\| = H$, we understand that the condition $V'_{(2.3)}(t, \phi, \varphi) \leq 0$ is satisfied in the case V' can be defined.

Then, for given initial value $\phi \in C_{H_1}$, $H_1 < H$, there exists a unique solution of (2.1).

Lemma 2.3. *(See [32] pp 190). Suppose that a continuous Lyapunov functional $V(t, \phi)$ exists, defined on $t \in \mathbf{R}^+$, $\|\phi\| < H$, $0 < H_1 < H$ which satisfies the following conditions:*

(i) $a(\|\phi\|) \leq V(t, \phi) \leq b(\|\phi\|)$, where $a(r)$ and $b(r)$ are continuous, increasing and positive,

(ii) $\dot{V}_{(2.1)}(t, \phi) \leq -c(\|\phi\|)$, where $c(r)$ is continuous and positive for $r \geq 0$,

then the zero solution of system (2.1) is uniformly asymptotically stable.

Lemma 2.4. *(See [5] pp 317). Let $V : \mathbf{R}^+ \times C \to \mathbf{R}$ be continuous and locally Lipschitz in ϕ. If*

(i) $W_0(|X_t|) \leq V(t, X_t) \leq W_1(|X_t|) + W_2\left(\int_{t-r(t)}^{t} W_3(X_t(s))ds\right)$ and

(ii) $\dot{V}_{(2.1)}(t, X_t) \leq -W_4(|X_t|) + N$, for some $N > 0$ where W_i $(i = 0, 1, 2, 3, 4)$ are wedges.

Then X_t of system 2.1 is uniformly bounded and uniformly ultimately bounded for bound M.

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3. Main Results

We will start with the following notations, let $x(t) = x$, $y(t) = y$, $\phi(x(t)) = \phi(x)$, $g(t, x(t-\tau(t)), x'(t-\tau(t)))x'(t) = g(\cdot)$ and $p(t, x(t-\tau(t)), x'(t-\tau(t))) = p(\cdot)$. Let (x_t, y_t) be any solution of the system (1.2), the continuously differentiable functional used in this investigation is $V = V(t, x_t, y_t)$ defined as

$$2V = \left[a^2 + b\phi(x)(1 + b\phi(x))\right]x^2 + (1 + b\phi(x))y^2 + 2axy$$

$$+\int_{-\tau(t)}^{0}\int_{t+s}^{t} \lambda y^2(\theta)\phi^{-2}(x(\theta))d\theta ds, \tag{3.1}$$

where the function ϕ is defined in Section 1, a, b, λ are positive constants and the value of λ will be determined later. Next, we state the main results as follows.

Theorem 3.1. *Further to the basic assumptions on the functions ϕ, g, h, τ and p, suppose that $a, b, \phi_0, \phi_1, \alpha, \beta, L$ and M are positive constants such that*

(i) $\phi_0 \leq \phi(x) \leq \phi_1$, $\phi'(x) \leq 0$ *for all* x;

(ii) $a \leq g(\cdot)$ *for all* $t \geq 0$, $x, y, x(t-\tau(t))$ *and* $y(t-\tau(t))$;

(iii) $bx \leq h(x) \leq Lx$ *for all* $x \neq 0$;

(iv) $\tau(t) \leq \alpha$, $\tau'(t) \leq \beta$, $0 < \beta < 1$ *where*

$$\alpha < \min\left\{\frac{b\phi_0}{L},\ \frac{2ab\phi_0(1-\beta)}{L[1 + a + b\phi_1 + 2(1-\beta)(1 + b\phi_1)]}\right\}; \tag{3.2}$$

(v) $|p(\cdot)| \leq M$, $0 < M < \infty$ *for all* $t \geq 0$, $x, y, x(t-\tau(t))$ *and* $y(t-\tau(t))$;

then the solution (x_t, y_t) of system (1.2) is uniformly bounded and uniformly ultimately bounded.

Remark 1. *We observed the following:*

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(i) *If* $\phi(x) = 1$, $g(\cdot) = a$, $h(t - \tau(t)) = bx$ *and* $p(\cdot) = 0$, *equation (1.1) reduces to linear constant coefficients differential equation*

$$x'' + ax' + bx = 0,$$

and conditions (i) to (v) of Theorem 3.1 specializes to the corresponding Routh-Hurwitz criteria $a > 0$ *and* $b > 0$.

(ii) *When* $\phi(x) = 1$, $\tau(t) = 0$ *and* $p(\cdot) = f(t)$, *equation (1.1) reduces to a special case discussed in [20], thus Theorem 3.1 includes and improves the results in [20].*

(iii) *Whenever* $\phi(x) = \phi(t)$, $g(\cdot) = a(t)f(x, x')$, $h(x(t - \tau(t))) = g(x)$ *and* $p(\cdot) = p(t, x, x')$, *equation (1.1) reduces to that discussed in [4], thus Theorem 3.1 improves the boundedness results in [4].*

(iv) *When* $\phi(x) = 1$, $g(\cdot) = f(x)$, $h(x(t - \tau(t))) = g(x)$ *and* $p(\cdot) = p(t, x, x')$, *equation (1.1) becomes that considered in [18], hence our results extend the results in [18].*

(v) *If* $g(\cdot) = g(t, x, x')$, $h(x(t - \tau(t))) = \varphi(t)h(x)$ *and* $p(\cdot) = p(t, x, x')$, *equation (1.1) coincides with (1.1) in [1], thus the results in [1] are contained in this work.*

In what follows we will state and prove a result that would be useful in the proof of Theorem 3.1 and the subsequent results.

Lemma 3.2. *Under the hypotheses of Theorem 3.1 there exist positive constants* $D_0 = D_0(a, b, \phi_0)$, $D_1 = D_1(a, b, \phi_1)$ *and* $D_2 = D_2(\phi_0, \lambda)$ *such that*

$$D_0(x^2(t)+y^2(t)) \le V(t, x_t, y_t) \le D_1(x^2(t)+y^2(t))+D_2 \int_{-\tau(t)}^{0} \int_{t+s}^{t} y^2(\theta)d\theta ds, \tag{3.3}$$

for all $t \ge 0$, x *and* y. *Furthermore, there exist positive constants* $D_3 = D_3(a, b, \alpha, \beta, L, \phi_1)$ *and* $D_4 = D_4(a, b, \phi_1)$ *such that*

$$V'_{(1.2)} \le -D_3(x^2(t) + y^2(t)) + D_4(|x(t)| + |y(t)|)|p(\cdot)|, \tag{3.4}$$

for all $t \ge 0$, x *and* y.

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Proof. Let (x_t, y_t) be any solution of system (1.2), from equation (3.1) for $x = 0 = y$, we have $V(t, 0, 0) = 0$ for all $t \geq 0$. Also the functional V defined in (3.1) can be recast in the form

$$2V = b\phi(x)[1+b\phi(x)]x^2+b\phi(x)y^2+(ax+y)^2+\int_{-\tau(t)}^{0}\int_{t+s}^{t}\lambda y^2(\theta)\phi^{-2}(x(\theta))d\theta ds. \tag{3.5}$$

In view of condition (i) of Theorem 3.1, $\phi(x) \neq 0$, and the fact that

$$\int_{-\tau(t)}^{0}\int_{t+s}^{t}\lambda y^2(\theta)\phi^{-2}(x(\theta))d\theta ds \geq 0,$$

for all $t \geq 0, x$ and y, there exists a positive constant δ_0 such that

$$V \geq \delta_0(x^2 + y^2), \tag{3.6}$$

for all $t \geq 0$, x and y where

$$\delta_0 := \frac{1}{2}\min\left\{b\phi_0(1 + b\phi_0) + \min\{1, a\},\ b\phi_0 + \min\{1, a\}\right\}.$$

In addition, from inequality (3.6) we find that

$$V(t, x_t, y_t) = 0 \quad \text{if and only if} \quad x^2 + y^2 = 0; \tag{3.7}$$

$$V(t, x_t, y_t) > 0 \quad \text{if and only if} \quad x^2 + y^2 \neq 0; \quad \text{and that} \tag{3.8}$$

$$V(t, x_t, y_t) \to +\infty \text{ as } x^2 + y^2 \to \infty. \tag{3.9}$$

Moreover, since $\phi_0 \leq \phi(x) \leq \phi_1$ for all x and the fact that the inequality $2xy \leq x^2 + y^2$ holds, there exist positive constants δ_1 and δ_2 such that

$$V \leq \delta_1(x^2 + y^2) + \delta_2\int_{-\tau(t)}^{0}\int_{t+s}^{t}y^2(\theta)d\theta ds, \tag{3.10}$$

for all $t \geq 0$, x and y, where

$$\delta_1 := \frac{1}{2}\max\{(1 + a)a + (1 + b\phi_1)b\phi_1,\ 1 + a + b\phi_1\}$$

and

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$$\delta_2 := \frac{\lambda}{2\phi_0^2}.$$

Inequalities (3.6) and (3.10) established the inequality (3.3) with δ_0, δ_1 and δ_2 equivalent to D_0, D_1 and D_2 respectively.

Next, the differentiation of the functional V defined by equation (3.1) with respect to the independent variable t along the solution path of system (1.2) after simplification is

$\mathrm{V}'_{(1.2)} = -\sum_{j=1}^{2} W_j + \sum_{j=3}^{4} W_j + \tau(t)\phi^{-2}(x)y^2 + [ax + (1 + b\phi(x))y]p(\cdot)$

$$- [1 - \tau'(t)]\lambda \int_{t-\tau(t)}^{t} \phi^{-2}(x(\theta))y^2(\theta)d\theta \tag{3.11}$$

where

$\mathrm{W}_1 := \frac{a}{2}\frac{h(x)}{x}x^2 + bg(\cdot)y^2;$

$W_2 := \frac{a}{4}\frac{h(x)}{x}x^2 + a[g(\cdot) - a]\phi^{-1}(x)xy + \frac{1}{2}\phi^{-1}(x)[g(\cdot) - a]y^2$

$+ \frac{a}{4}\frac{h(x)}{x}x^2 + (1 + b\phi(x))\left[\frac{h(x)}{x} - b\right]xy + \frac{1}{2}\phi^{-1}(x)[g(\cdot) - a]y^2;$

$W_3 := \frac{1}{2}\phi'(x)y[b(2b + \phi^{-1}(x))x^2 + b\phi^{-1}(x)y^2];$ and

$W_4 := [ax + (1 + b\phi(x))y] \int_{t-\tau(t)}^{t} h'(x(s))y(s)\phi^{-1}(x(s))ds.$

Now from conditions (ii) and (iii) of Theorem 3.1, we find that

$$W_1 \geq \frac{1}{2}ab(x^2 + y^2),$$

for all $t \geq 0, x$ and y. Using conditions (i), (ii) and (iii) of Theorem 3.1 in W_2 we have

$$W_2 \geq \frac{a}{4}\left[bx^2 + \frac{4}{\phi_1}(g(\cdot) - a)xy + \frac{2}{a\phi_1}(g(\cdot) - a)y^2\right]$$

$$+ \frac{1}{4}\left[abx^2 + 4(1 + b\phi_0)\left(\frac{h(x)}{x} - b\right)xy + \frac{2}{\phi_1}(g(\cdot) - a)y^2\right], \tag{3.12}$$

for all $t \geq 0, x$ and y. Employing estimates

$$\left[\frac{4}{\phi_1}\left(g(\cdot) - a\right)\right]^2 < \frac{b}{a\phi_1}\left(g(\cdot) - a\right) \text{ and } \left[4(1 + b\phi_0)\left(\frac{h(x)}{x} - b\right)\right]^2 < \frac{ab}{\phi_1}\left(g(\cdot) - a\right)$$

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inequality (3.12) becomes

$$W_2 \geq \frac{a}{4}\left[\sqrt{b}|x| - \sqrt{2a^{-1}\phi_1^{-1}(g(\cdot)-a)}|y|\right]^2$$
$$+\frac{1}{4}\left[\sqrt{ab}|x| - \sqrt{2\phi_1^{-1}(g(\cdot)-a)}|y|\right]^2 \geq 0,$$

for all $t \geq 0, x$ and y. Moreover, from hypothesis (i) and let $y > 0$ it follows from W_3 that

$$y\left[b(2b+\phi^{-1}(x))x^2 + b\phi^{-1}(x))y^2\right] \geq (2b+\phi_1^{-1})x^2 + b\phi_1^{-1}y^2 \geq 0 \tag{3.13}$$

for all $t \geq 0, x$ and y. Using inequality (3.13) and $\phi'(x) \leq 0$ for all x in W_3, we obtain

$$W_3 \leq 0$$

for all $t \geq 0, x$ and y. Finally, $\phi_0 \leq \phi(x) \leq \phi_1$ for all x, $h'(x) \leq L$ for all x and the inequality $xy \leq \frac{1}{2}(x^2+y^2)$ imply that

$$W_4 \leq \frac{aL}{2\phi_0}\tau(t)x^2 + \frac{L}{2\phi_0}(1+b\phi_1)\tau(t)y^2 + \frac{L}{2\phi_0}(1+a+b\phi_1)\int_{t-\tau(t)}^{t} y^2(\theta)d\theta,$$

for all $t \geq 0, x$ and y. Inserting estimate W_j $(j = 1, \cdots, 4)$ in equation (3.11), we obtain

$$\begin{aligned} \mathrm{V}'_{(1.2)} \leq\ & -\tfrac{ab}{2}(x^2+y^2) - \left[[1-\tau'(t)]\phi_1^{-2}\lambda - \tfrac{L}{2\phi_0}(1+a+b\phi_1)\right]\int_{t-\tau(t)}^{t} y^2(\theta)d\theta \\ & + \left[\tfrac{L}{2\phi_0}(1+b\phi_1) + \phi_1^{-2}\lambda\right]\tau(t)y^2 + \tfrac{aL}{2\phi_0}\tau(t)x^2 + \left[a|x| + (1+b\phi_1)|y|\right]|p(\cdot)| \end{aligned} \tag{3.14}$$

for all $t \geq 0, x$ and y. Furthermore, $\tau(t) \leq \alpha$, $\tau'(t) \leq \beta$, $\beta \in (0,1)$ and choose

$$\lambda := 2^{-1}(1-\beta)^{-1}\phi_0^{-1}\phi_1^2 L(1+a+b\phi_1) > 0 \tag{3.15}$$

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there exist positive constants δ_3 and δ_4 such that

$$V'_{(1.2)} \leq -\delta_3(x^2 + y^2) + \delta_4(|x| + |y|)|p(\cdot)|, \tag{3.16}$$

for all $t \geq 0, x$ and y where

$$\delta_3 := \frac{1}{2}\min\left\{a\left(b - \frac{\alpha L}{\phi_0}\right), \quad ab - \frac{L}{\phi_0}\left[1 + b\phi_1 + (1-\beta)^{-1}(1 + a + b\phi_1)\right]\alpha\right\}$$

(3.17)

and

$$\delta_4 := \max\{a, \quad 1 + b\phi_1\}.$$

Inequality (3.16) satisfies the inequality (3.4) with δ_3 and δ_4 equivalent to D_3 and D_4 respectively. This completes the proof of Lemma 3.2 □ Next, we will give the prove of Theorem 3.1, using some of the estimates of Lemma 3.2.

Proof of Theorem 3.1 Let (x_t, y_t) be any solution of system (1.2). From inequalities (3.10) and (3.10) hypothesis (i) of Lemma 2.4 holds. Furthermore, using assumption (v) of Theorem 3.1 in estimate (3.16), noting that $|x| < 1 + x^2$, there exist positive constants δ_5 and δ_6 such that

$$V'_{(1.2)} \leq -\delta_5(x^2 + y^2) + \delta_6 \tag{3.18}$$

where $\delta_5 := \delta_3 - \delta_4 M$ with $\delta_4 M$ choosing sufficiently small and $\delta_6 := 2\delta_4 M$. Inequality (3.18) satisfies condition (ii) of Lemma 2.4. Thus by Lemma 2.4 the solution (x_t, y_t) of system (1.2) is uniformly bounded and uniformly ultimately bounded. This completes the proof of Theorem 3.1. □

Next, if the forcing term $p(\cdot)$ of equation (1.1) is replaced by a function $p_1(t)$ where $p_1(t)$ is defined on $\mathbf{R}^+$, we obtain a special case of equation (1.1) as

$$[\phi(x)x']' + g(\cdot)x' + h(x(t - \tau(t))) = p_1(\cdot). \tag{3.19}$$

Equation (3.19) is equivalent to system of first order equations

$$\begin{aligned} x' &= \; y\phi^{-1}(x), \\ y' &= \; p_1(t) + \int_{t-\tau(t)}^{t} h'(x(s))y(s)\phi^{-1}(x(s))ds - h(x) - g(\cdot)y\phi^{-1}(x), \end{aligned}$$

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(3.20)

where the functions g, h and ϕ are defined in Section 1. We obtain the following result.

Theorem 3.3. *If assumptions (i) to (iv) of Theorem 3.1 hold and assumption (v) is replaced by boundedness of the function* $p_1(t)$, *then the solution* (x_t, y_t) *of system (3.20) is uniformly bounded and uniformly ultimately bounded.*

Proof. Let (x_t, y_t) be any solution of system (3.20), the remaining part of the prove is similar to the proof of Theorem 3.1 hence it is omitted. This completes the proof of Theorem 3.3. □

Furthermore, if the function $p(\cdot)$ of equation (1.1) is replaced by $p_2(t, x, x')$ defined on $\mathbf{R}^+ \times \mathbf{R}^2$, we have the following equation

$$[\phi(x)x']' + g(\cdot)x' + h(x(t-\tau(t))) = p_2(t, x, x'). \tag{3.21}$$

Equation (3.21) can be written as system of first order differential equations

$$x' = y\phi^{-1}(x),$$
$$y' = p_2(t, x, x') + \int_{t-\tau(t)}^{t} h'(x(s))y(s)\phi^{-1}(x(s))ds - h(x) - g(\cdot)y\phi^{-1}(x),$$
(3.22)

and we have following result.

Theorem 3.4. *If the forcing term* p_2 *of system (3.22) is bounded and assumptions (i) to (iv) of Theorem 3.1 hold, then the solution* (x_t, y_t) *of system (3.22) is uniformly bounded and uniformly ultimately bounded.*

Proof. Let (x_t, y_t) be any solution of system (3.22), the remaining part of the prove is similar to the proof of Theorem 3.1 hence it is omitted. This completes the proof of Theorem 3.4. □

Next, if $p(\cdot)$ of equation 1.1 is zero we have the following special case

$$[\phi(x)x']' + g(\cdot)x' + h(x(t-\tau(t))) = 0. \tag{3.23}$$

Equation (3.23) as system of first order differential equations are as follow

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$$x' = y\phi^{-1}(x),$$

$$y' = \int_{t-\tau(t)}^{t} h'(x(s))y(s)\phi^{-1}(x(s))ds - h(x) - g(\cdot)y\phi^{-1}(x), \tag{3.24}$$

and we have following result.

Theorem 3.5. *If assumptions (i) to (iv) of Theorem 3.1 hold, then the trivial solution of system (3.24) is uniformly asymptotically stable.*

Proof. Let (x_t, y_t) be any solution of system (3.24), from equation (3.1) we have $V(t, 0, 0) = 0$ for all $t \geq 0$. Moreover, the inequalities (3.6) and (3.10) satisfy condition (i) of Lemma 2.3. Also if $p(\cdot) = 0$, estimate (3.16) becomes

$$V'_{3.21} \leq -\delta_3(x^2 + y^2) \tag{3.25}$$

for all $t \geq 0, x$ and y where δ_3 is defined in (3.17). Inequality (3.25) establish assumption (ii) of Lemma 2.3, hence by Lemma 2.3 the trivial solution of system (3.24) is uniformly asymptotically stable. This completes the proof of Theorem 3.5. □

Next, we will state and proofs existence and uniqueness results of the solutions of system (1.2).

Theorem 3.6. *If assumptions of Theorem 3.1 hold, then there exists a periodic solution of system 1.2 of period ω.*

Proof. Let (x_t, y_t) be any solution of system (1.2), from inequalities (3.6), (3.10) and estimate (3.9), assumption (i) of Lemma 2.1 hold. Moreover, using hypothesis (v) of Theorem 3.6 and inequality $|x| + |y| \leq 2^{1/2}(x^2 + y^2)^{1/2}$ in estimate (3.16) there exist positive constants δ_7 and δ_8 such that

$$V'_{(1.2)} \leq -\delta_7(x^2 + y^2) \leq 0 \tag{3.26}$$

for all $t \geq 0, x$ and y provided that

$$(x^2 + y^2)^{1/2} \geq \delta_8$$

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where $\delta_7 := \frac{1}{2}\delta_3$ and $\delta_8 := 2^{3/2}\delta_3^{-1}M$. Inequality (3.26) satisfies assumption (ii) of Lemma 2.1, thus by Lemma 2.1 the periodic solution of system (1.2) exists and is of period ω. This completes the proof of Theorem 3.6. □

Theorem 3.7. *If assumptions of Theorem 3.1 are satisfied, then there exists a unique solution of system (1.2).*

Proof. Let (x_t, y_t) be any solution of system (1.2), in view of estimates (3.6), (3.7), (3.9) and (3.26), assumptions of Lemma 2.2 hold, thus by Lemma 2.2 solution of system (1.2) is unique. This completes the proof of Theorem 3.7. □

4. Examples

Example 4.1. *Consider the second order delay differential equation*

$$\left[\frac{4(1+e^x)}{(3+4e^x)}x'\right]' + \left[\frac{2+t^2+x^2(t-\tau(t))+x'^2(t-\tau(t))}{1+t^2+x^2(t-\tau(t))+x'^2(t-\tau(t))}\right]x' + \left[\frac{2+x^2(t-\tau(t))}{1+x^2(t-\tau(t))}\right]\times$$

$$x(t-\tau(t)) = \frac{2+t^2+\cos[x(t-\tau(t))]+\cos[x'(t-\tau(t))]}{2[t^2+\cos[x(t-\tau(t))]+\cos[x'(t-\tau(t))]]}. \tag{4.1}$$

Equation (4.1) is equivalent to

$$x' = \frac{(3+4e^x)}{4(1+e^x)}y,$$

$$\begin{aligned} y' = {} & \frac{2+t^2+\cos[x(t-\tau(t))]+\cos[y(t-\tau(t))\phi^{-1}(x(t-\tau(t)))]}{2[t^2+\cos[x(t-\tau(t))]+\cos[y(t-\tau(t))\phi^{-1}(x(t-\tau(t)))]]} \\ & + \int_{t-\tau(t)}^{t} \frac{2+x^2(s)+x^4(s)}{(1+x^2(s))^2}ds - \frac{x(2+x^2)}{1+x^2} \\ & - \left[\frac{2+t^2+x^2(t-\tau(t))+y^2(t-\tau(t))\phi^{-2}(x(t-\tau(t)))}{1+t^2+x^2(t-\tau(t))+y^2(t-\tau(t))\phi^{-2}(x(t-\tau(t)))}\right]\times \\ & y(t-\tau(t))\phi^{-1}(x(t-\tau(t))). \end{aligned} \tag{4.2}$$

Comparing systems (1.2) and (4.2) we find that:

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(i) *The function*

$$\phi(x) = \frac{4(1+e^x)}{(3+4e^x)} = 1 + \frac{1}{3+4e^x}.$$

Since

$$0 \leq \frac{1}{3+4e^x} \leq 1$$

for all x, *it follows that*

$$1 = \phi_0 \leq \phi(x) \leq \phi_1 = 2$$

for all x. *Furthermore,*

$$\phi'(x) = \frac{-4e^x}{(3+4e^x)^2} \leq 0$$

for all x. *The paths of* $\phi(x)$ *and its derivative,* $\phi'(x)$, *are shown in Figure 1*

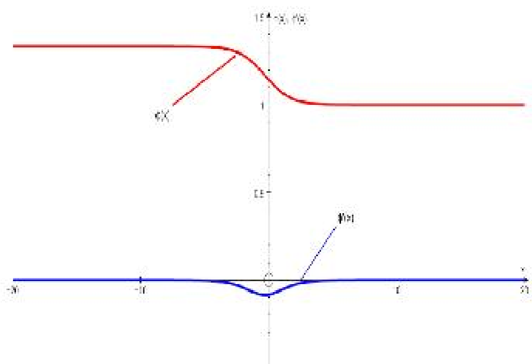

FIGURE 1. Functions $\phi(x(t))$, and $\phi'(x(t))$

(ii) *The function*

$$g(\cdot) = \frac{2+t^2+x^2(t-\tau(t))+y^2(t-\tau(t))\phi^{-2}(x(t-\tau(t)))}{1+t^2+x^2(t-\tau(t))+y^2(t-\tau(t))\phi^{-2}(x(t-\tau(t)))}$$

$$= 1 + \frac{1}{1+t^2+x^2(t-\tau(t))+y^2(t-\tau(t))\phi^{-2}(x(t-\tau(t)))}$$

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Since the fraction

$$\frac{1}{1+t^2+x^2(t-\tau(t))+y^2(t-\tau(t))\phi^{-2}(x(t-\tau(t)))} \geq 0$$

for all $t \geq 0$, $x(t-\tau(t))$ *and* $y(t-\tau(t))$, *it follows that*

$$g(\cdot) \geq a = 1$$

for all $t \geq 0$, $x(t-\tau(t))$ *and* $y(t-\tau(t))$.

(iii) The function

$$h(x) := \frac{x(2+x^2)}{1+x^2} = x + \frac{x}{1+x^2}$$

or

$$\frac{h(x)}{x} = 1 + \frac{1}{1+x^2}.$$

Since

$$0 \leq H(x) = \frac{1}{1+x^2} \leq 1$$

for all x *it follows that*

$$1 = b \leq \frac{h(x)}{x} \leq L = 2$$

for all $x \neq 0$.

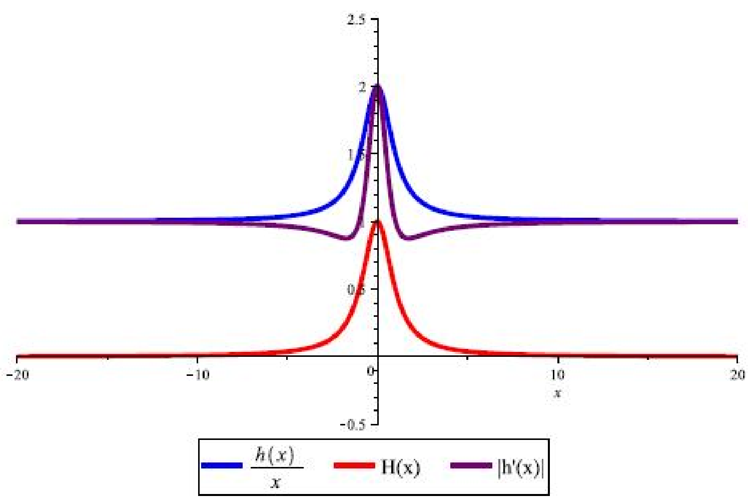

FIGURE 2. The behaviour of the functions $h(x)/x, H(x)$ and $|h'(x)|$

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Furthermore, the derivative of the function h *with respect to* x *is*

$$h'(x) = \frac{2 + x^2 + x^4}{(1 + x^2)^2} = 1 + \frac{1 - x^2}{(1 + x^2)^2},$$

Noting that

$$\frac{1 - x^2}{(1 + x^2)^2} \leq 1$$

for all x*, it follows that*

$$h'(x) = 1 + \frac{1 - x^2}{(1 + x^2)^2} \leq 2$$

and

$$|h'(x)| \leq L = 2$$

for all x*. The behaviour of the functions* $h(x)/x, H(x)$ *and* $h'(x)$ *are shown in Figure 4*

(iv) *Using estimates (i) to (iii) of Example 4.1 with* $\beta = \frac{1}{2}$*, inequality (3.2) and equation (3.15) become*

$$\alpha < \min\left\{\frac{1}{2}, \ \frac{1}{22}\right\} = \frac{1}{22} \ \textit{and} \ \ \lambda = 32 > 0$$

respectively.

(v) *The function*

$$p(\cdot) = \frac{2 + t^2 + \cos[x(t - \tau(t))] + \cos[y(t - \tau(t))\phi^{-1}(x(t - \tau(t)))]}{2[t^2 + \cos[x(t - \tau(t))] + \cos[y(t - \tau(t))\phi^{-1}(x(t - \tau(t)))]]}$$

$$= \frac{1}{2} + \frac{1}{t^2 + \cos[x(t - \tau(t))] + \cos[y(t - \tau(t))\phi^{-1}(x(t - \tau(t)))]}$$

It is not difficult to show that

$$|p(\cdot)| \leq M = \frac{3}{2}$$

for all $t \geq 0$*,* $x(t - \tau(t))$ *and* $y(t - \tau(t))$*.*

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From items (i) to (v) of Example 4.1, the assumption of Theorem 3.1, Theorem 3.6 and Theorem 3.7 hold, thus by Theorem 3.1, Theorem 3.6 and Theorem 3.7 the solution (x_t, y_t) *of system (4.2).*

(i) is uniformly bounded and uniformly ultimately bounded;

(ii) possess a periodic solution of period ω*; and*

(iii) is unique.

Also, if $p(\cdot) = 0$ *in system (4.2), items (i) to (iv) of Example 4.1 are equivalent to hypotheses (i) to (iv) of Theorem 3.5, then by Theorem 3.5 the trivial solution of system 4.2 is uniformly asymptotically stable.*

Example 4.2. *Consider also, the second order delay differential equation*

$$\left[\frac{5+6e^{(2x+1)}}{2+3e^{(2x+1)}}x'\right]' + \frac{3+\cos[tx(t-\tau(t))]+\cos[x'(t-\tau(t))]}{2[1+\cos[tx(t-\tau(t))]+\cos[x'(t-\tau(t))]]}$$
$$+\frac{4x(t-\tau(t))+7x^3(t-\tau(t))+x(t-\tau(t))\cos(2x(t-\tau(t)))}{2+7x^2(t-\tau(t))}$$
$$= \frac{3+4t+2|x(t-\tau(t))|+2|x'(t-\tau(t))|}{1+2t+|x(t-\tau(t))|+|x'(t-\tau(t))|}. \tag{4.3}$$

Equation (4.3) in its equivalent form is

$$x' = \frac{2+3e^{(2x+1)}}{5+6e^{(2x+1)}}y,$$
$$y' = \frac{3+4t+2|x(t-\tau(t))|+2|y(t-\tau(t))\phi^{-1}(x(t-\tau(t)))|}{1+2t+|x(t-\tau(t))|+|y(t-\tau(t))\phi^{-1}(x(t-\tau(t)))|}$$
$$+\int_{t-\tau(t)}^{t}\frac{(7x^2-2)\cos 2x+2(7x^2+2)(x\sin 2x-7x^2-2)}{(7x^2+2)^2}(s)ds$$
$$-\frac{4x+7x^3+x\cos 2x}{2+7x^2}$$
$$-\frac{3+\cos\left[tx\left(t-\tau\left(t\right)\right)\right]+\cos\left[y\left(t-\tau\left(t\right)\right)\phi^{-1}\left(x\left(1-\tau\right)\right)\right]}{2\left[1+\cos\left[tx\left(1-\tau\left(t\right)\right)\right]+\cos\left[y\left(t-\tau\left(t\right)\right)\phi^{-1}\left(x\left(t-\tau\left(t\right)\right)\right)\right]\right]}. \tag{4.4}$$

Comparing system (1.2) with system (4.4), we find that

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(i) *the function*

$$\phi(x) := \frac{5 + 6e^{(2x+1)}}{2 + 3e^{(2x+1)}} = 2 + \frac{1}{2 + 3e^{(2x+1)}}.$$

Let

$$\Phi(x) := \frac{1}{2 + 3e^{(2x+1)}}.$$

It is not difficult to show that

$$2 = \phi_0 \leq \phi(x) \leq \phi_1 = 3$$

for all x. *Moreover,*

$$\phi'(x) = -\frac{6e^{2x+1}}{(2 + 3e^{(2x+1)})^2} \leq 0$$

for all x. *The behaviour of the functions* $\phi(x)$ *and* $\phi'(x)$ *are shown in Figure 3*

(ii) *the function*

$$g(\cdot) := \frac{3 + \cos[tx(t-\tau(t))] + \cos[y(t-\tau(t))\phi^{-1}(x(t-\tau(t))]}{2[1 + \cos[tx(t-\tau(t))] + \cos[y(t-\tau(t))\phi^{-1}(x(t-\tau(t))]]}$$

$$= \frac{1}{2} + \frac{1}{1 + \cos[tx(t-\tau(t))] + \cos[y(t-\tau(t))\phi^{-1}(x(t-\tau(t))]}$$

from where we obtain

$$g(\cdot) \geq a = \frac{1}{2},$$

for all $t \geq 0, x(t-\tau(t))$ *and* $y(t-\tau(t))$.

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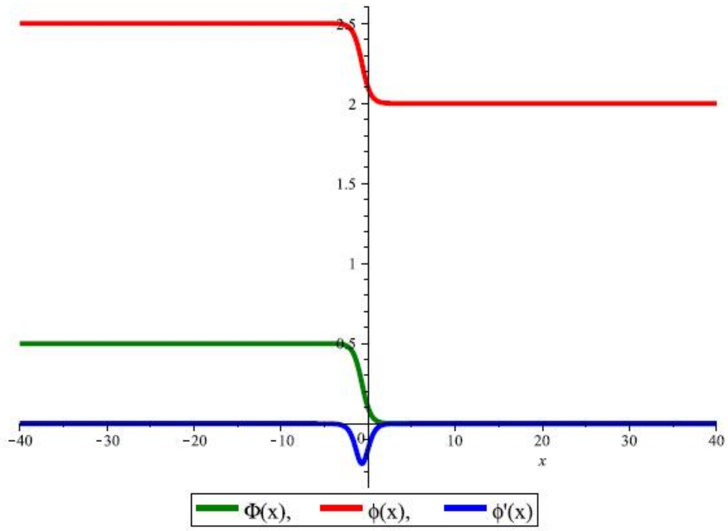

FIGURE 3. The behaviour of the functions $\Phi(x), \phi(x)$ and $\phi'(x)$.

(iii) the function

$$h(x) := \frac{4x + 7x^3 + x\cos 2x}{2 + 7x^2} = 2x + \frac{x\cos 2x}{2 + 7x^2}.$$

This can be recast in the form

$$\frac{h(x)}{x} = 2 + \frac{\cos 2x}{2 + 7x^2} = 2 + H(x),$$

where

$$H(x) = \frac{\cos 2x}{2 + 7x^2}.$$

It is not difficult to show that

$$-\frac{1}{2} \leq H(x) \leq \frac{1}{2}$$

for all x. It follows that

$$\frac{3}{2} = b \leq \frac{h(x)}{x} \leq L = \frac{5}{2}$$

for all $x \neq 0$. In addition,

$$|h'(x)| \leq L = \frac{5}{2}$$

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for all x*. Alternatively,*

$$h'(x) = \frac{(7x^2-2)\cos 2x + 2(7x^2+2)(x\sin 2x - 7x^2 - 2)}{(7x^2+2)^2}$$

$$= 2 - \frac{(7x^2-2)\cos 2x + 2x(7x^2+2)x\sin 2x}{(7x^2+2)^2}$$

It follows that

$$|h'(x)| \le L = \frac{5}{2}$$

for all x*. The behaviour of the functions* $h(x)/x, H(x)$ *and* $|h'(x)|$ *are shown in Figure 3*

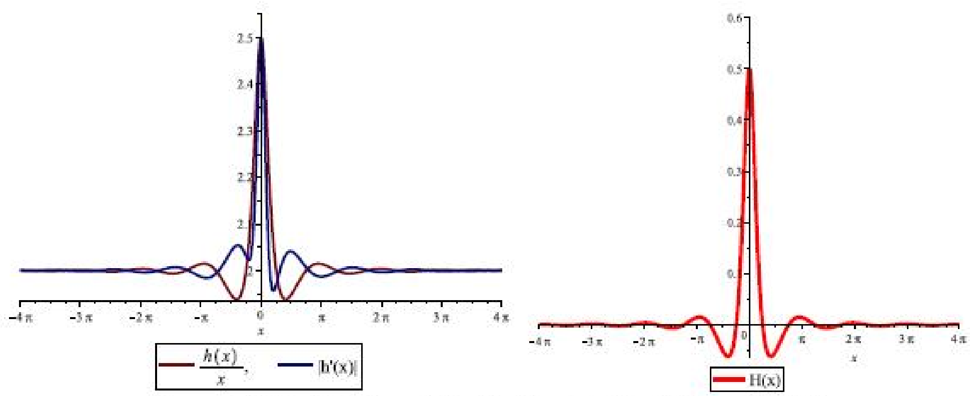

FIGURE 4. The behaviour of the functions $h(x)/x, H(x)$ and $|h'(x)|$.

(iv) *From items (i) - (iii) of Example 4.2, choose* $\beta = \frac{1}{2}$ *inequality 3.2 and equation (3.15) become*

$$\alpha < \min\left\{\frac{6}{5}, \frac{6}{175}\right\} = \frac{6}{175}, \quad \textit{and} \quad \lambda = 270 > 0$$

respectively.

(v) *Finally, the function*

$$p(\cdot) := \frac{3 + 4t + 2|x(t-\tau(t))| + 2|y(t-\tau(t))\phi^{-1}(x(t-\tau(t)))|}{1 + 2t + |x(t-\tau(t))| + |y(t-\tau(t))\phi^{-1}(x(t-\tau(t)))|}$$

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$$= 2 + \frac{1}{1 + 2t + |x(t - \tau(t))| + |y(t - \tau(t))\phi^{-1}(x(t - \tau(t)))|}$$

where we obtain

$$|p(\cdot)| \leq M = 3$$

for all $t \geq 0$, $x(t - \tau(t))$ *and* $y(t - \tau(t))$.

From items (i) to (v) of Example **??**, *assumptions of Theorem 3.1, Theorem 3.6 and Theorem 3.7 hold, thus by Theorem 3.1, Theorem 3.6 and Theorem 3.7 the solution* (x_t, y_t) *of system (4.4)*

(i) is uniformly bounded and uniformly ultimately bounded;

(ii) possess a periodic solution of period ω; *and*

(iii) is unique.

Also, if $p(\cdot) = 0$ *in system (4.4), items (i) to (iv) of Example 4.2 satisfy the assumptions of Theorem 3.5, then by Theorem 3.5 the trivial solution of system 4.4 is uniformly asymptotically stable.*

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