Transnational Journal of Mathematical Analysis and Applications
Vol. 8, Issue 1, 2020, Pages 17-23
ISSN 2347-9086
Published Online on December 21, 2020

http://jyotiacademicpress.org

NEW DISCOVERIES ON THE FINITE *p*-GROUPS OF ORDER $2^{(n+6)}$

S. A. ADEBISI[1], M. OGIUGO[2] and M. ENIOLUWAFE[3]

[1]Department of Mathematics
Faculty of Science
University of Lagos
Nigeria
e-mail: adesinasunday@yahoo.com

[2]Department of Mathematics
School of Science
Yaba College of Technology
Lagos
Nigeria
e-mail: ekpenogiugo@gmail.com

[3]Department of Mathematics
Faculty of Science
University of Ibadan
Nigeria
e-mail: michael.enioluwafe@gmail.com

2020 Mathematics Subject Classification: Primary 20D15, 20E28, 20F18, 20N25, 20K27; Secondary 08A05, 08A72, 20E34, 20K01, 20K35.

Keywords and phrases: finite *p*-groups, abelian group, fuzzy subsets, fuzzy subgroups, isomorphic, inclusion-exclusion principle, maximal subgroups, nilpotent group, explicit formulae.

Received October 14, 2020; Revised November 11, 2020

Abstract

The finite nilpotent groups can now be formed in various dimensions. As such, results up to two dimensions are now obtainable. In this paper, the fuzzy subgroups of the nilpotent product of two abelian subgroups of orders 2^n and 64. Here, the integers $n > 6$ have been successfully considered and the derivation for the explicit formulae for its number distinct fuzzy subgroups were calculated.

1. Introduction

From inception, several methods, techniques and approaches were used for the classification of which some are obtainable in [6], and, for example, the natural equivalence relation was introduced in [10]. In this work, an essential role in solving counting problems is played by adopting the "Inclusion-Exclusion Principle". The process leads to some recurrence relations from which the solutions are finally computed with ease. In the process of our computation, the use of GAP (Group Algorithm and Programming) was actually applied.

2. Basic Definitions and Terms

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are satisfied:

(i) $\lambda(ab) \geq \min\{\lambda(a), \lambda(b)\}, \forall a, b \in G;$

(ii) $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G.$

And, since $(a^{-1})^{-1} = a,$ we have that $\lambda(a^{-1}) = \lambda(a),$ for any $a \in G.$

Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$ (Marius [7]). Define by $M_1, M_2, \ldots, M_t,$ the maximal subgroups of $G,$ and denote by $h(G)$ the number of chains of subgroups of G which ends in $G.$

Theorem (Marius [7])**.** *The set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.*

We define the level subset: $\lambda G_{\beta} = \{a \in G / \lambda(a) \geq \beta\}$ for each $\beta \in [0, 1]$. The fuzzy subgroups of a finite p-group G are thus, characterized, based on these subsets. In the sequel, λ is a fuzzy subgroup of G if and only if its level subsets are subgroups in G. This theorem gives a link between $FL(G)$ and $L(G)$, the classical subgroup lattice of G.

Moreover, some natural relations on $S(G)$ can also be used in the process of classifying the fuzzy subgroups of a finite q-group G (see [9] and [10]). One of them is defined by: $\lambda \sim \gamma$ iff $(\lambda(a) > \lambda(b) \Longleftrightarrow \nu(a) > \nu(b)$, $\forall a, b \in G)$. Also, two fuzzy subgroups λ, γ of G are said to be distinct if $\lambda \nsim v$.

As a result of this development, let G be a finite p-group and suppose that $\lambda : G \longrightarrow [0, 1]$ is a fuzzy subgroup of G. Put $\lambda(G) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 < \beta_2 > \cdots > \beta_k$. Then, ends in G is determined by λ.

$$\lambda G_{\beta 1} \subset \lambda G_{\beta 2} \subset \cdots \subset \lambda G_{\beta_k} = G. \quad \text{(a)}$$

Also, we have that

$\lambda(a) = \beta_t \Longleftrightarrow t = \max\{r / a \in \lambda G_{\beta r}\} \Longleftrightarrow a \in \lambda G_{\beta t} \setminus \lambda G_{\beta t-1}$, for any $a \in G$ and $t = 1, \ldots, k$, where by convention, set $\lambda G_{\beta 0} = \varnothing$.

3. The Techniques

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \tag{1}$$

In [8], (#) was used to obtain the explicit formulas for some positive integers n.

Theorem $(^*)$ (Marius [10]). *The number of distinct fuzzy subgroups of a finite p-group of order* p^n *which have a cyclic maximal subgroup is*:

(i) $h(\mathbb{Z}_{p^n}) = 2^n$;

(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}(2 + (n-1)p)$.

Proposition A (see [4]). *Suppose that* $G = \mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$.

Then

$$h(G) = \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$$

$$+ \frac{1}{3}(2^{n+1})\sum_{k=1}^{n-5}[((n-k)^3 + 12(n-k)^2 + 17(n-k) - 24).$$

Proposition B (see [5]). *Suppose that* $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^n}$.

Then

$$h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})] + \sum_{k=1}^{n-6} 2^{k+1}\left(h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}})\right).$$

4. Computation for $G = \mathbb{Z}_{64} \times \mathbb{Z}_{2^n}$

Suppose that $G = \mathbb{Z}_{64} \times \mathbb{Z}_{64}$ Then, $h(G) = 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6})$.

If $G = \mathbb{Z}_{64} \times \mathbb{Z}_{2^7}$, then $h(G) = 2[h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^6}) + h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^7}) - h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6})$.

For $G = \mathbb{Z}_{64} \times \mathbb{Z}_{2^8}$, we have that

$$h(G) = 2[h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^7}) + h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^8}) - h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^7}).$$

Also, if $G = \mathbb{Z}_{64} \times \mathbb{Z}_{2^9}$, then

$$h(G) = 2[h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^8}) + h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^9}) - h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^8}).$$

Now, let $G = \mathbb{Z}_{64} \times \mathbb{Z}_{2^n}$, then

$$\begin{aligned}
h(G) &= 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) - 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^{n-1}}) \\
&= 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-2}}) \\
&\quad + 4h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^{n-2}}) \\
&= 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + 4h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-2}}) \\
&\quad - 8h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-3}}) + 8h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^{n-3}}) \\
&= 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + 4h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-2}}) \\
&\quad + 8h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-3}}) - 16h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^{n-4}})
\end{aligned}$$

$$= 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}})$$

$$+ 4\sum_{k=1}^{t-2} 2^{k-1}\left(h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1-k}})\right) - 2^t h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-t}})$$

$$+ 2^t h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^{n-t}}),$$

where, $n - t = 6$, implying that $t = n - 6$.

Therefore

$$h(G) = 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + 4\sum_{k=1}^{n-8} 2^{k-1}\left(h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1-k}})\right)$$

$$- 2^{n-6} h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6}) + 2^{n-6} h(\mathbb{Z}_{64} \times \mathbb{Z}_{2^6})$$

$$= 2^{n-4}[h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5})]$$

$$+ 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}})$$

$$+ 4\sum_{k=1}^{n-8} 2^{k-1}\left(h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1-k}})\right),$$

where, $h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^n})$ is found using Proposition B. □

5. Conclusion

We have been able to successfully classified and the number of distinct fuzzy subgroups for the abelian structure formed from two nilpotent subgroups of orders 64 and 2^n, respectively where $n \geq 6$. This has been made possible by a comprehensive analysis and the application of GAP (Group Algorithms and Programming, Version 4.8.7; https: www.gap-system.org).

References

[1] S. A. Adebisi, M. Ogiugo and M. Enioluwafe, Computing the number of distinct fuzzy subgroups for the nilpotent p-group of $D_2n \times C_4$, International Journal of Mathematical Combinatorics 1 (2020), 86-89.

[2] S. A. Adebisi, M. Ogiugo and M. Enioluwafe, Determining the number of distinct fuzzy subgroups for the abelian structure: $Z_4 \times Z_{2_{n-1}}$, $n > 2$, Journal of the Nigerian Association of Mathematical Physics 11 (2020), 5-6.

[3] S. A. Adebisi, M. Ogiugo and M. Enioluwafe, The fuzzy subgroups for the abelian structure $Z_8 \times Z_{2^n}$, $n > 2$, Journal of the Nigerian Mathematical Society 39(2) (2020), 167-171.

[4] S. A. Adebisi, M. Ogiugo and M. Enioluwafe, Fuzzy subgroups for the Cartesian product of the abelian structure $\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$, $n > 3$, Journal of The Nigerian Mathematical Society (Submitted).

[5] S. A. Adebisi, M. Ogiugo and M. Enioluwafe, Fuzzy subgroups for the abelian structure $(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}})$, Nigerian Journal of Mathematics and Applications (Submitted).

[6] M. Mashinchi and M. Mukaidono, A classification of fuzzy subgroups, Ninth Fuzzy System Symposium, Sapporo, Japan (1992), 649-652.

[7] Tarnauceanu Marius, Classifying fuzzy subgroups for a class of finite p-groups, All Cuza Univ., Iasi, Romania (2013), 30-39.

[8] Tarnauceanu Marius, Classifying fuzzy subgroups of finite nonabelian groups, Iranian Journal of Fuzzy Systems 9(4) (2012), 31-41.

DOI: https://doi.org/10.22111/IJFS.2012.131

[9] L. Bentea and M. Tarnauceanu, A note on the number of fuzzy subgroups of finite groups, Analele Stiintifice ale Universitatii All Cuza din Iasi - Matematica 54(1) (2008), 209-220.

[10] Tarnauceanu Marius and L. Bentea, On the number of fuzzy subgroups of finite Abelian groups, Fuzzy Sets and Systems 159(9) (2008), 1084-1096.

DOI: https://doi.org/10.1016/j.fss.2007.11.014

■