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The Modular Nilpotent Group $M_{p^n} \times C_p$ for $p > 2$

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Abstract: In this paper, the classification of finite p-groups is extended to the modular nilpotent group of the form $M_{p^n} \times C_p$ in which, p is greater than 2.

Key Words: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups.

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§1. Introduction

The following properties for the fuzzy subgroups of G were known:

(1) The level sets of a fuzzy subset of a finite set form a chain;

(2) λ is a fuzzy subgroup of G iff its level sets are subgroups of G';

(3) The relation $\sim$ is an equivalence relation on fuzzy subgroups of G,where for fuzzy subgroups μ, ν of G , $\mu \sim \nu$ iff $\forall x, y \in G, (\mu(x) > \mu(y) iff \nu(x) > \nu(y))$.

§2. Preliminaries

Suppose that $(G, \cdot, e)$ is a group with identity e. Let $S(G)$ denote the collection of all fuzzy subsets of G. An element $\lambda \in S(G)$ is said to be a fuzzy subgroup of G if the following two conditions are satisfied:

(*i*) $\lambda(ab) \geq \in \{\lambda(a), \lambda(b)\}, \ \forall\, a, b \in G$;

(*ii*) $\lambda(a^{-1} \geq \lambda(a)$ for any $a \in G$.

And, since $(a^{-1})^{-1} = a$, we have that $\lambda(a^{-1}) = \lambda(a)$, for any $a \in G$. Also, by this notation and definition, $\lambda(e) = \sup \lambda(G)$. [Marius [1]].

Now, concerning the subgroups, the set $FL(G)$ possessing all fuzzy subgroups of G forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of G.

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In what follows, the method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \cdots, M_t$ are the maximal subgroups of G. Let $h(G)$ denote the number of chains of subgroups of G which ends in G. The method of computing $h(G)$ is based on the application of the inclusion-exclusion principle. If A is the set of chains in G of type

$$C_1 \subset C_2 \subset \cdots \subset C_r = G$$

and A' represents the set of chains of A' which are contained in M_r, $r = 1, \cdots, t$. Then, we have

$$\begin{aligned} |A| &= 1 + |A'| = \left| \bigcup_{r=1}^{t} A_r \right| \\ &= 1 + \sum_{r=1}^{t} |A_r| - \sum_{1 \le r_1 \le r_2 \le t} |A_{r_1} \cap A_{r_2}| + \cdots + (-1)^{t-1} \left| \bigcap_{r=1}^{t} A_r \right| \end{aligned}$$

Observe that, for every $1 \le w \le t$ and $1 \le r_1 < r_2 < \cdots < r_w \le t$, the set $\bigcap_{i=1}^{w} A_{r_i}$ consists of all chains of A' which are included in $\bigcap_{i=1}^{w} M_{r_i}$. We have that

$$\left| \bigcap_{i=1}^{w} A_{r_i} \right| = 2h \left(\bigcap_{i=1}^{w} M_{r_i} \right)^{-1}$$

Therefore,

$$\begin{aligned} |A| &= 1 + \sum_{r=1}^{t} (2h(M_r) - 1) - \sum_{1 \le r_1 < r_2 \le t} (2h(M_{r_1} \cap M_{r_2}) - 1) \\ &\quad + \cdots + (-1)^{t-1} \left(2h \left(\bigcap_{r=1}^{t} M_r \right) - 1 \right) \\ &= 2 \left(\sum_{r=1}^{t} h(M_r) - \sum_{1 \le r_1 < r_2 \le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h \left(\bigcap_{r=1}^{t} M_r \right) \right) + C, \end{aligned}$$

where, the constant C can be determined by

$$\begin{aligned} C &= 1 + \sum_{r=1}^{t} (-1) - \sum_{1 \le r_1 < r_2 \le t} (-1) + \cdots + (-1)^{t-1}(-1) \\ &= (1-1)^t = 0 \end{aligned}$$

and we have that

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\leq r_1<r_2\leq t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) \qquad (c)$$

In [2], the equality (c) was used to obtain the explicit formulas of $h(D_{2n})$ for some positive integers n.

Theorem 2.1 *The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is:*

(1) $h(\mathbb{Z}_{p^n}) = 2^n$;

(2) $h(D_{2^n}) = 2^{2n-1}$;

(3) $h(\varphi_{2^n}) = 2^{2n-2}$;

(4) $h(S_{2^n}) = 3.2^{2n-3}$;

(5) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

§3. The Fuzzy Subgroup for the Nilpotent Group of the Form: $M_{p^n} \times C_p$

Recall that the case for $p = 2$ was already handled in [3]. Now, for $p > 2$ and, of course, p is a prime, we consider this case as follows.

3.1 The Derivation of $h(M_{p^n} \times C_p)$ for $p > 2$

We begin with the case $p = 3$ and $n = 3$ where

$$\begin{aligned} M_{3^3} &= \langle x, y | x^9 = y^3 = 1,\ y^{-1}xy = x^4\rangle \\ &= \left\{\begin{array}{l} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, \\ y, y^2, xy, xy^2, x^2y, x^2y^2, x^3y, x^3y^2, x^4y, \\ x^4y^2, x^5y, x^5y^2, x^6y, x^6y^2, x^7y, x^7y^2, x^8y, x^8y^2 \end{array}\right\} \end{aligned}$$

and

$$M_{3^3} \times C_3 = \left\{\begin{array}{l} 1, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, \\ y, y^2, xy, xy^2, x^2y, x^2y^2, x^3y, x^3y^2, x^4y, \\ x^4y^2, x^5y, x^5y^2, x^6y, x^6y^2, x^7y, x^7y^2, x^8y, x^8y^2 \end{array}\right\} \times \{1, a, a^2\}$$

$$
= \left\{ \begin{array}{l}
(1,1),(1,a),(1,a^2),(x,1),(x,a),(x,a^2),(x^2,1),(x^2,a),(x^2,a^2), \\
(x^3,1),(x^3,a),(x^3,a^2),(x^4,1),(x^4,a),(x^4,a^2),(x^5,1),(x^5,a), \\
(x^5,a^2),(x^6,1),(x^6,a),(x^6,a^2),(x^7,1),(x^7,a),(x^7,a^2), \\
(x^8,1),(x^8,a),(x^8,a^2),(y,1),(y,a),(y,a^2),(y^2,1),(y^2,a),(y^2,a^2) \\
(xy,1),(xy,a),(xy,a^2),(xy^2,1),(xy^2,a),(xy^2,a^2),(x^2y,1) \\
(x^2y,a),(x^2y,a^2),(x^2y^2,1),(x^2y^2,a),(x^2y^2,a^2),(x^3y,1) \\
(x^3y,a),(x^3y,a^2),(x^3y^2,1)(x^3y^2,a),(x^3y^2,a^2) \\
(x^4y,1),(x^4y,a),(x^4y,a^2),(x^4y^2,1),(x^4y^2,a) \\
(x^4y^2,a^2),(x^5y,1),(x^5y,a),(x^5y,a^2) \\
(x^5y^2,1),(x^5y^2,a),(x^5y^2,a^2),(x^6y,1), \\
(x^6y,a),(x^6y,a^2),(x^6y^2,1),(x^6y^2,a), \\
(x^6y^2,a^2),(x^7y,1),(x^7y,a),(x^7y,a^2) \\
(x^7y^2,1),(x^7y^2,a),(x^7y^2,a^2),(x^8y,1), \\
(x^8y,a),(x^8y,a^2),(x^8y^2,1),(x^8y^2,a),(x^8y^2,a^2)
\end{array} \right\}
$$

Lemma 3.1(Berkovich,[3]) *Let G be a group of order p^n.*

(i) If A is a subgroup of G of order p^k and $k < m < n$, then, the number of subgroups of G of order p^m containing $A \equiv 1(\mathrm{mod} p)$.

(ii) If G is a noncyclic group of order p^n, $1 < m < n-1$, then,

$$S_m(G) \in \{1+p, 1+p+p^2\},$$

where $S_m(G)$ is the number of subgroups of order p^m in G.

By Lemma 3.1, let $\mathcal{M}$ be the collection of all the maximal subgroups of G. Then, the set

$$|\mathcal{M}| = 1+p+p^2$$

and there exists 13 distinct maximal subgroups for $M_{3^3} \times C_3$. These maximal subgroups are generated by

$$
\left\{ \begin{array}{l}
\langle (1,a),(x,1)\rangle, \langle (y,1),(x,1)\rangle, \langle (y,a),(x,1)\rangle, \langle (y,a),(x,a)\rangle, \\
\langle (y,1),(x,a)\rangle, \langle (y,a^2),(x,a)\rangle, \langle (xy^2,1),(1,a)\rangle, \\
\langle (xy,1),(1,a)\rangle, \langle (y,1),(x,a^2)\rangle, \langle (y,a),(x,a^2)\rangle, \\
\langle (y,a^2),(x,1)\rangle, \langle (y,1),(1,a),(x^3,1)\rangle \text{ and } \langle (y,a^2),(x,a^2)\rangle
\end{array} \right\}.
$$

We therefore have

$$M_1 = \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(x,1),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1), \\ (x^7,1),(x^8,1),(x,a),(x,a^2),(x^2,a),(x^2,a^2),(x^3,a),(x^3,a^2),(x^4,a), \\ (x^4,a^2),(x^5,a),(x^5,a^2),(x^6,a),(x^6,a^2),(x^7,a),(x^7,a^2),(x^8,a),(x^8,a^2) \end{array} \right\},$$

$$M_2 = \left\{ \begin{array}{l} (1,1),(x,1),(y,1),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1),(x^7,1), \\ (x^8,1),(y^2,1),(xy,1),(xy^2,1),(x^2y,1),(x^2y^2,1),(x^3y,1),(x^3y^2,1), \\ (x^4y,1),(x^4y^2,1),(x^5y,1),(x^5y^2,1),(x^6y,1),(x^6y^2,1),(x^7y,1), \\ (x^7y^2,1),(x^8y,1),(x^8y^2,1) \end{array} \right\},$$

$$M_3 = \left\{ \begin{array}{l} (1,1),(y,a),(x,1),(y^2,a^2),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1), \\ (x^7,1),(x^8,1),(xy,a),(xy^2,a^2),(x^2y,a),(x^2y^2,a^2),(x^3y,a), \\ (x^3y^2,a^2),(x^4y,a),(x^4y^2,a^2),(x^5y,a),(x^5y^2,a^2),(x^6y,a),(x^6y^2,a^2), \\ (x^7y,a),(x^7y^2,a^2),(x^8y,a),(x^8y^2,a^2) \end{array} \right\},$$

$$M_4 = \left\{ \begin{array}{l} (1,1),(x,a),(y,a),(x^2,a^2),(x^3,1),(x^4,a),(x^5,a^2),(x^6,1),(x^7,a), \\ (x^8,a^2),(y^2,a^2),(xy,a^2),(xy^2,1),(x^2y,1),(x^2y^2,a),(x^3y,a),(x^3y^2,a^2), \\ (x^4y,a^2),(x^4y^2,1),(x^5y,1),(x^5y^2,a),(x^6y,a),(x^6y^2,a^2),(x^7y,a^2), \\ (x^7y^2,1),(x^8y,1),(x^8y^2,a) \end{array} \right\},$$

$$M_5 = \left\{ \begin{array}{l} (1,1),(x,a),(y,1),(x^2,a^2),(x^3,1),(x^4,a),(x^5,a^2),(x^6,1),(x^7,a), \\ (x^8,a^2),(y^2,1),(xy,a),(xy^2,a),(x^2y,a^2),(x^2y^2,a^2),(x^3y,1),(x^3y^2,1), \\ (x^4y,a),(x^4y^2,a),(x^5y,a^2),(x^5y^2,a^2),(x^6y,1),(x^6y^2,1),(x^7y,a), \\ (x^7y^2,a),(x^8y,a^2),(x^8y^2,a^2) \end{array} \right\},$$

$$M_6 = \left\{ \begin{array}{l} (1,1),(1,a),(y,1),(x^3,1),(1,a^2),(y^2,1),(x^6,1),(y,a),(y^2,a),(y,a^2), \\ (y^2,a^2),(x^3y,1),(x^3y^2,1),(x^6y,1),(x^6y^2,1),(x^3,a),(x^3,a^2),(x^6,a), \\ (x^6,a^2),(x^3y^2,a^2),(x^3y^2,a),(x^3y,a^2),(x^3y,a),(x^6y,a^2),(x^6y,a), \\ (x^6y^2,a^2),(x^6y^2,a) \end{array} \right\},$$

$$M_7 = \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(xy^2,1),(x^5y,1),(x^3,1),(x^4y^2,1),(x^8y,1),(x^6,1), \\ (x^7y^2,1),(x^2y,1),(xy^2,a),(x^5y,a),(x^3,a),(x^4y^2,a),(x^8y,a),(x^6,a), \\ (x^7y^2,a),(x^2y,a),(xy^2,a^2),(x^5y,a^2),(x^3,a^2),(x^4y^2,a^2),(x^8y,a^2), \\ (x^6,a^2),(x^7y^2,a^2),(x^2y,a^2) \end{array} \right\},$$

$$M_8 = \left\{ \begin{array}{l} (1,1),(1,a),(1,a^2),(xy,1),(x^8y^2,1),(x^3,1),(x^4y,1),(x^2y^2,1), \\ (x^6,1),(x^7y,1),(x^5y^2,1),(xy,a),(x^8y^2,a),(x^3,a),(x^4y,a), \\ (x^2y^2,a),(x^6,a),(x^7y,a),(x^5y^2,a),(xy,a^2),(x^8y^2,a^2),(x^3,a^2), \\ (x^4y,a^2),(x^2y^2,a^2),(x^6,a^2),(x^7y,a^2),(x^5y^2,a^2) \end{array} \right\},$$

$$M_9 = \left\{ \begin{array}{l} (1,1),(y,1),(y^2,1),(x,a^2),(x^2,a),(x^3,1),(x^4,a^2),(x^5,a),(x^6,1), \\ (x^7,a^2),(x^8,a),(xy,a^2),(x^2y,a),(x^3y,1),(x^4y,a^2),(x^5y,a),(x^6y,1), \\ (x^7y,a^2),(x^8y,a),(xy^2,a^2),(x^2y^2,a),(x^3y^2,1),(x^4y^2,a^2),(x^5y^2,a), \\ (x^6y^2,1),(x^7y^2,a^2),(x^8y^2,a) \end{array} \right\},$$

$$M_{10} = \left\{ \begin{array}{l} (1,1),(y,a),(y^2,a^2),(x,a^2),(x^2,a),(x^3,1),(x^4,a^2),(x^5,a),(x^6,1), \\ (x^7,a^2),(x^8,a),(xy,1),(x^2y,a^2),(x^3y,a),(x^4y,1),(x^5y,a^2),(x^6y,a), \\ (x^7y,1),(x^8y,a^2),(xy^2,a),(x^2y^2,1),(x^3y^2,a^2),(x^4y^2,a),(x^5y^2,1), \\ (x^6y^2,a^2),(x^7y^2,a),(x^8y^2,1) \end{array} \right\},$$

$$M_{11} = \left\{ \begin{array}{l} (1,1),(y,a^2),(y^2,a),(x,1),(x^2,1),(x^3,1),(x^4,1),(x^5,1),(x^6,1),(x^7,1) \\ (x^8,1),(xy,a^2),(x^2y,a^2),(x^3y,a^2),(x^4y,a^2),(x^5y,a^2),(x^6y,a^2), \\ (x^7y,a^2),(x^8y,a^2),(xy^2,a),(x^2y^2,a),(x^3y^2,a),(x^4y^2,a),(x^5y^2,a), \\ (x^6y^2,a),(x^7y^2,a),(x^8y^2,a) \end{array} \right\},$$

$$M_{12} = \left\{ \begin{array}{l} (1,1),(y,a^2),(y^2,a),(x,a),(x^2,a^2),(x^3,1),(x^4,1),(x^5,a^2), \\ (x^6,1),(x^7,a),(x^8,a^2),(xy,1),(x^2y,a),(x^3y,a^2),(x^4y,1),(x^5y,a), \\ (x^6y,a^2),(x^7y,1),(x^8y,a),(xy^2,a^2),(x^2y^2,1),(x^3y^2,a),(x^4y^2,a^2), \\ (x^5y^2,1),(x^6y^2,a),(x^7y^2,a^2),(x^8y^2,1) \end{array} \right\},$$

$$M_{13} = \left\{ \begin{array}{l} (1,1),(y,a^2),(y^2,a),(x,a^2),(x^2,a),(x^3,1),(x^4,a^2),(x^5,a),(x^6,1), \\ (x^7,a^2),(x^8,a),(xy,a),(x^2y,1),(x^3y,a^2),(x^4y,a),(x^5y,1),(x^6y,a^2), \\ (x^7y,a),(x^8y,1),(xy^2,1),(x^2y^2,a^2),(x^3y^2,a),(x^4y^2,1),(x^5y^2,a^2), \\ (x^6y^2,a),(x^7y^2,1),(x^8y^2,a^2) \end{array} \right\}.$$

By the equality (c), we have

$$\begin{aligned} \frac{1}{2}h(M_{3^3}\times C_3) &= (3h(\mathbb{Z}_3\times\mathbb{Z}_{3^2}) + 9h(M_{3^3}) + h(\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3)) \\ &\quad -[24h(\mathbb{Z}_3\times\mathbb{Z}_3) + 54h(\mathbb{Z}_{3^2})] + [234h(\mathbb{Z}_3) + 36h(\mathbb{Z}_{3^2}) \\ &\quad +16h(\mathbb{Z}_3\times\mathbb{Z}_3)] - [702h(\mathbb{Z}_3) + 4h(\mathbb{Z}_3\times\mathbb{Z}_3) + 9h(\mathbb{Z}_{3^2}) \\ &\quad +1287h(\mathbb{Z}_3) - 1716h(\mathbb{Z}_3) + 1716h(\mathbb{Z}_3) \\ &\quad -1287h(\mathbb{Z}_3) + 715h(\mathbb{Z}_3) - 286h(\mathbb{Z}_3) \\ &\quad +78h(\mathbb{Z}_3) - 13h(\mathbb{Z}_3) + h(\mathbb{Z}_3) \\ &= 3h(\mathbb{Z}_3\times\mathbb{Z}_{3^2}) + 9h(M_{3^3}) + h(\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3) \\ &\quad -27h(\mathbb{Z}_{3^2}) - 12h(\mathbb{Z}_3\times\mathbb{Z}_3) + 27h(\mathbb{Z}_3). \end{aligned}$$

Therefore,

$$h(M_{3^3}\times C_3) = 420 + 2h(\mathbb{Z}_3\times\mathbb{Z}_3\times\mathbb{Z}_3) = 420 + 2(158) = 736.$$

3.2 Determination of $h(M_{5^3}\times C_5)$

Following a careful analysis and subsequent operations on the maximal subgroups, we have an estimate given by:

$$\begin{aligned} \frac{1}{2}h(M_{5^3}\times C_5) &= [ph(\mathbb{Z}_p\times\mathbb{Z}_{p^2}) + h(\mathbb{Z}_p\times\mathbb{Z}_p\times\mathbb{Z}_p) + p^2h(M_{p^3})] \\ &\quad - \left[(p+1)h(\mathbb{Z}_p\times\mathbb{Z}_p)\begin{pmatrix} p+1 \\ 2 \end{pmatrix} + \begin{pmatrix} p+1 \\ 2 \end{pmatrix}\cdot p^2h(\mathbb{Z}_{p^2})\right] \\ &\quad + \left[(p+1)\begin{pmatrix} p+1 \\ 3 \end{pmatrix}h(\mathbb{Z}_p\times\mathbb{Z}_p) + p^2\begin{pmatrix} p+1 \\ 3 \end{pmatrix}h^2(\mathbb{Z}_{p^2})\right. \\ &\quad + \left[\left(\begin{pmatrix} 1+p+p^2 \\ 3 \end{pmatrix} - (1+p+p^2)\begin{pmatrix} p+1 \\ 3 \end{pmatrix}\right)h(\mathbb{Z}_p)\right] \\ &\quad - \left[(p+1)\begin{pmatrix} p+1 \\ 4 \end{pmatrix}h(\mathbb{Z}_p\times\mathbb{Z}_p) + \begin{pmatrix} p+1 \\ 4 \end{pmatrix}\cdot p^2h(\mathbb{Z}_{p^2})\right. \end{aligned}$$

$$
\begin{aligned}
&+\left[\left[\binom{1+p+p^2}{4} - (1+p+p^2)\binom{p+1}{4}\right] h(\mathbb{Z}_p)\right] \\
&+\left[(p+1)\binom{p+1}{5} h(\mathbb{Z}_p \times \mathbb{Z}_p) + p^2 \cdot \binom{p+1}{5} h(\mathbb{Z}_{p^2})\right. \\
&+\left[\left[\binom{1+p+p^2}{5} - (1+p+p^2)\binom{p+1}{5}\right] h(\mathbb{Z}_p)\right] \\
&-\left[(p+1)h(\mathbb{Z}_p \times \mathbb{Z}_p) + p^2 h(\mathbb{Z}_{p^2}) - \left(31 - \binom{31}{6}\right) h(\mathbb{Z}_p)\right] \\
&+\left[\binom{31}{7} - \binom{31}{8} + \cdots + \binom{31}{29} - \binom{31}{30} + 1\right] h(\mathbb{Z}_p)
\end{aligned}
$$

3.3 Determination of $h(M_{p^n} \times C_p)$

In general,

$$
\begin{aligned}
\frac{1}{2}h(M_{p^n} \times C_p) &= ph(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) + p^2 h(M_{p^n}) \\
&\quad + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}\mathbb{Z}_{p^{n-2}}) - p(p+1)h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) \\
&\quad - p^3 h(\mathbb{Z}_{p^{n-1}}) + p^3 h(\mathbb{Z}_{p^{n-2}}) \\
&= p(p+1)(2^{n-1}))np - p + 2) - p(p+1)(2^{n-2})(np - 2p + 2) \\
&\quad + p^3(2^{n-2} - 2^{n-1}] + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) \\
&= 2^{n-2}[p(p+1)(np+2) - p^3] + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}})
\end{aligned}
$$

Therefore,

$$h(M_{p^n} \times C_p) = 2^{n-1}[p(p+1)(np+2) - p^3] + 2h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}).$$

$$
\begin{aligned}
h(M_{p^n} \times C_p) &= 2^{n-1}[p(p+1)(np+2) - p^3] \\
&\quad + 2^{n-1}[(3n-5)p + (n^2-5)p^2 + (n^2-5n+8)p^3 + 4] - 4p^3 \\
&= 2^{n-1}[(p^2+p^3)n^2 + (3p+p^2-4p^3)n + (7p^3-3p^2-3p+4)] - 4p^3 \\
&= 2^{n-1}[p^2(1+p)n^2 + p(3+p-4p^2)n + (7p^3-3p^2-3p+4)] - 4p^3.
\end{aligned}
$$

Therefore, for modular finite p-groups

$$h(M_{p^n} \times C_p) = 2^{n-1}[p^2(1+p)n^2 + p(3+p-4p^2)n + (7p^3-3p^2-3p+4)] - 4p^3$$

for $p > 2$.

Theorem 3.2 *Let $G = M_{p^n} \times C_p$, the modular nilpotent group formed by taking the cartesian product of the modular p-group of order p^n and a cyclic group of order p, where p is a prime. Then, the number of distinct fuzzy subgroups of G for $n > 4$ is given by*

$$h(G) = 2^{n-1} \times \big[p^2(1+p)n^2 + p(3+p-4p^2)n + (7p^3 - 3p^2 - 3p + 4)\big] - 4p^3$$

for $p > 2$.

Proof For all values of p, there exist only one maximal subgroup which is isomorphic to the Abelian type $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}$. p of the maximal subgroups are isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}$, while p^2 of them are isomorphic to M_{p^n}.

If we put these values into equation(c), we have as follows:

$$\begin{aligned}
\frac{1}{2}h(M_{p^n} \times C_p) &= ph(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) + p^2 h(M_{p^n}) \\
&\quad + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) - p(p+1)h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) \\
&\quad - p^3 h(\mathbb{Z}_{p^{n-1}}) + p^3 h(\mathbb{Z}_{p^{n-2}}) \\
&= p(p+1)(2^{n-1})(np - p + 2) - p(p+1)(2^{n-2})(np - 2p + 2) \\
&\quad + p^3(2^{n-2} - 2^{n-1}] + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}) \text{ (by Theorem 2.1((1) and (5))} \\
&= 2^{n-2}[p(p+1)(np+2) - p^3] + h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}).
\end{aligned}$$

Therefore,

$$h(M_{p^n} \times C_p) = 2^{n-1}[p(p+1)(np+2) - p^3] + 2h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}}).$$

Here, recurrence relation was used for the purpose of $h(\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_{p^{n-2}})$. We now have

$$\begin{aligned}
h(M_{p^n} \times C_p) &= 2^{n-1}[p(p+1)(np+2) - p^3] \\
&\quad + 2^{n-1}[(3n-5)p + (n^2-5)p^2 + (n^2 - 5n + 8)p^3 + 4] - 4p^3 \\
&= 2^{n-1}[p^2(1+p)n^2 + p(3+p-4p^2)n + (7p^3 - 3p^2 - 3p + 4)] - 4p^3
\end{aligned}$$

for $p > 2$ □

References

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