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A VECTOR MATRIX APPROACH OF COUNTING CYCLIC QUOTIENTS OF SOME ABELIAN P-GROUPS

Michael Enioluwafe[1]
Department of Mathematics, University of Ibadan, Ibadan, Nigeria
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

We determine in this paper, the precise number of cyclic quotients of Abelian p-groups of exponent p^i and rank $r > 1, \quad i = 1, 2, \ldots, n$ for all natural numbers n.

MIRAMARE – TRIESTE
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[1] Regular Associate of ICTP. maalawo@yahoo.com

1 INTRODUCTION

The mathematical motivation for this paper is as follows:
Let π be a finite Abelian group, R a commutative Noetherian ring, $G_*(\Lambda)$ the Quillen K-theory of the category of finitely-generated Λ-modules, for any ring Λ with identity. In [2], D. L. Webb established the formula

$$G_n(\mathbb{Z}\pi) \simeq \bigoplus_{\rho \,\in\, X(\pi)} G_n(\mathbb{Z} < \rho >) \ , \quad n \geq 0$$

where $\mathbb{Z} < \rho >$ denotes the ring of fractions $\mathbb{Z}(\rho)[1/|\rho|]$ obtained by inverting $|\rho|$, $\mathbb{Z}(\rho)$ denotes the quotient of the group ring $\mathbb{Z}\rho$ by the $|\rho|^{-th}$ cyclotomic polynomial $\Phi_{|\rho|}$ evaluated at a generator of ρ (the ideal factored out is independent of the choice of generator for ρ), $|.|$ denotes cardinality and $X(\pi)$ the set of cyclic quotients of π.

A natural problem is that of computing $G_n(\mathbb{Z}\pi)$ as explicitly as possible and from the formula above, it is desirable to know the number of cyclic quotients of π.
The object of this paper is to establish the precise number of cyclic quotients of π, for

$$\pi \ := \underbrace{\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n}_{r-times}, \quad n \geq 1, \quad r > 1$$

The results of the cases $n = 1$ and 2 have been completed and appears in [1]. The organization of the paper is as follows:

Section 2, which is the main body of the work, is devoted to a proof of the following generalized result.

Theorem M **:**

Let

$$\pi \ := \underbrace{\mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j}_{r-times}, \quad r > 1, \ j \ \in \ \{1, 2, ..., n\}, \quad p$$

a prime number and γ is a subgroup of π. Then the number of the cyclic factor groups π/γ up to isomorphism, such that $|\pi/\gamma| = p^j$ for all j summed to n, is $(\frac{p^r-1}{p-1})(\frac{p^{n(r-1)}-1}{p^{r-1}-1})$.

Section 3 is devoted to the conclusion and a proof of the following useful result:

Lemma E **:**

Let

$$\pi \ := \underbrace{\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n}_{r-times}, \quad r > 1, \quad n \ \text{ a positive integer}, \ \ p$$

a prime number and γ is a subgroup of π. Then the number of the cyclic factor groups π/γ up to isomorphism, such that $|\pi/\gamma| = p^n$, is $p^{(n-1)(r-1)}(\frac{p^r-1}{p-1})$.

2 MAIN BODY

In this paper, we need the following fundamental definition.

Definition: **(Fundamental)**

Let

$$\pi := \underbrace{\mathbb{Z}/p^i \oplus \mathbb{Z}/p^i \oplus \cdots \oplus \mathbb{Z}/p^i}_{r-times}, \quad i \geq 1, \quad r > 1, \quad p$$

a prime number and γ *a subgroup of* π *of order* p^{ir-i}*, then we define a subgroup base for* γ *as* $(r-i)$, r*-tuples generating* γ*. This can be represented as* $(r-i)$*-rows of an* $r \times r$*-matrix whose rows generate* π*.*

In this section, we first establish the following:

Lemma E **:**

Let

$$\pi := \underbrace{\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n}_{r-times}, \quad r > 1, \quad n \text{ a positive integer}, \quad p$$

a prime number and γ *is a subgroup of* π*. Then the number of the cyclic factor groups* π/γ *up to isomorphism, such that* $|\pi/\gamma| = p^n$, is $p^{(n-1)(r-1)}(\frac{p^r-1}{p-1})$.

Proof:

Let

$$\pi := \underbrace{\mathbb{Z}/p^n \oplus \mathbb{Z}/p^n \oplus \cdots \oplus \mathbb{Z}/p^n}_{r-times}, \quad r > 1, \ n \ \text{ a positive integer and } \ p$$

a prime number.

Then the required cyclic quotients are realized in n number of cases as follows:

Case 1:

We define

$\mathbb{Z}/p^n \cong \mathbb{Z}^*_{p^n} \ :=< a >$,

$\epsilon_\kappa \in \{a^l\}, \quad 0 \leq l \leq p^n - 1$

and applying the fundamental definition given above, we obtain the following set of subgroup base representations in $r \times r-$ matrices:

$$\mathcal{A} = \left\{ \begin{pmatrix} a^{p^n} & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & a & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & a & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & \epsilon_\kappa & 1 & \dots & 1 & 1 & 1 \\ 1 & a^{p^n} & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & a & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a & 1 & 1 & \dots & 1 & 1 & \epsilon_\kappa \\ 1 & a & 1 & \dots & 1 & 1 & \epsilon_\kappa \\ 1 & 1 & a & \dots & 1 & 1 & \epsilon_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & \epsilon_\kappa \\ 1 & 1 & 1 & \dots & 1 & a & \epsilon_\kappa \\ 1 & 1 & 1 & \dots & 1 & 1 & a^{p^n} \end{pmatrix} \right\}$$

Thus applying a counting rule on set $\mathcal{A}$ yields a total sum of cyclic quotients π/γ for which $|\pi/\gamma| = p^n$ as:

$$1 + p^n + (p^n)^2 + \dots + (p^n)^{r-3} + (p^n)^{r-2} + (p^n)^{r-1}$$

That is,

$$\frac{p^{nr} - 1}{p^n - 1}, \quad \text{for any prime } p \text{ and any integer } r > 1. \ \square$$

Next, consider

Case 2:

In this case, we define

$\mathbb{Z}/p^n \cong \{\mathbb{Z}^*_{p^{n-1}}, \mathbb{Z}^*_p\} :=< a >,$

$\epsilon_\alpha \in \{a^i\}, \quad 1 \le i \le p^{n-1}, \quad g\ c\ d(i, p^{n-1}) = 1,$

$\epsilon_\beta \in \{a^i\}, \quad 1 \le i \le p, \quad g\ c\ d(i, p) = 1,$

$\epsilon_\gamma \in \{a^k\}, \quad 0 \le k \le p^{n-1} - 1,$

$\epsilon_\kappa \in \{a^l\}, \quad 0 \le l \le p - 1$

and applying our fundamental definition together with a counting rule we form the following sets of subgroup base representations in $r \times r-$ matrices with their respective results:

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} a^{p^{n-1}} & \epsilon_\beta & 1 & \dots & 1 & 1 & 1 \\ 1 & a^p & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & a & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a^{p^{n-1}} & 1 & \epsilon_\beta & \dots & 1 & 1 & 1 \\ 1 & a & \epsilon_\kappa & \dots & 1 & 1 & 1 \\ 1 & 1 & a^p & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a^{p^{n-1}} & 1 & 1 & \dots & 1 & 1 & \epsilon_\beta \\ 1 & a & 1 & \dots & 1 & 1 & \epsilon_\kappa \\ 1 & 1 & a & \dots & 1 & 1 & \epsilon_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & \epsilon_\kappa \\ 1 & 1 & 1 & \dots & 1 & a & \epsilon_\kappa \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \right\}$$

This generates a total sum of cyclic quotients:

$$(p-1) + p(p-1) + \cdots + p^{r-2}(p-1),$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} a^p & 1 & 1 & \dots & 1 & 1 & \epsilon_\alpha \\ 1 & a & 1 & \dots & 1 & 1 & \epsilon_\gamma \\ 1 & 1 & a & \dots & 1 & 1 & \epsilon_\gamma \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & \epsilon_\gamma \\ 1 & 1 & 1 & \dots & 1 & a & \epsilon_\gamma \\ 1 & 1 & 1 & \dots & 1 & 1 & a^{p^{n-1}} \end{pmatrix}, \begin{pmatrix} a^p & 1 & 1 & \dots & 1 & \epsilon_\alpha & 1 \\ 1 & a & 1 & \dots & 1 & \epsilon_\gamma & 1 \\ 1 & 1 & a & \dots & 1 & \epsilon_\gamma & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & \epsilon_\gamma & 1 \\ 1 & 1 & 1 & \dots & 1 & a^{p^{n-1}} & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a^p & \epsilon_\alpha & 1 & \dots & 1 & 1 & 1 \\ 1 & a^{p^{n-1}} & 1 & \dots & 1 & 1 & 1 \\ 1 & 1 & a & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix} \right\}$$

This generates a total sum of cyclic quotients:
$(p^{n-1})^{r-2}(p^{n-1} - p^{n-2}) + (p^{n-1})^{r-3}(p^{n-1} - p^{n-2}) + \cdots + (p^{n-1} - p^{n-2})$.

Continuing with this rule we finally consider the following set of subgroup bases:

$$\mathcal{B}_t = \left\{ \begin{pmatrix} a & 1 & \epsilon_\kappa & \dots & \epsilon_\gamma & 1 & 1 \\ 1 & a & \epsilon_\kappa & \dots & \epsilon_\gamma & 1 & 1 \\ 1 & 1 & a^p & \dots & \epsilon_\alpha & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^{p^{n-1}} & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a & \epsilon_\kappa & \epsilon_\gamma & \dots & 1 & 1 & 1 \\ 1 & a^p & \epsilon_\alpha & \dots & 1 & 1 & 1 \\ 1 & 1 & a^{p^{n-1}} & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a & \epsilon_\gamma & \epsilon_\kappa & \dots & 1 & 1 & 1 \\ 1 & a^{p^{n-1}} & \epsilon_\beta & \dots & 1 & 1 & 1 \\ 1 & 1 & a^p & \dots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a & 1 & 1 \\ 1 & 1 & 1 & \dots & 1 & a & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix} \right\}$$

and obtain a sum of number of cyclic quotients as:

$$(p^{n-1})^{r-3}(p^{n-1} - p^{n-2})p^2 + (p^{n-1})(p^{n-1} - p^{n-2})p + \cdots + p(p-1)(p^{n-1}),$$

where

$$|\mathcal{B}_1| + |\mathcal{B}_2| + \cdots + |\mathcal{B}_t| = \ r(r-1)$$

Continuing in this way with the other cases, we next consider, the following last case.

<u>Case $n-1$:</u>
In this case, we define

$$\mathbb{Z}/p^n \ \cong \ \underbrace{\{\mathbb{Z}^*_{p^{n-r+2}}, \mathbb{Z}^*_p, \cdots, \mathbb{Z}^*_p\}}_{(r-1)-\text{terms}} \ := < a >,$$

$\epsilon_\alpha \ \in \ \{a^i\}, \quad 1 \leq i \leq p^{n-r+2}, \quad g\ c\ d(i, p^{n-r+2}) = 1,$
$\epsilon_\beta \ \in \ \{a^i\}, \quad 1 \leq i \leq p, \quad g\ c\ d(i, p) = 1,$
$\epsilon_\gamma \ \in \ \{a^k\}, \quad 0 \leq k \leq p^{n-r+2} - 1,$
$\epsilon_\kappa \ \in \ \{a^l\}, \quad 0 \leq l \leq p - 1,$
and similarly, applying our fundamental definition together with counting rule we form the following sets of subgroup base representations in $r \ \times r-$ matrices with their respective results:

$$\mathcal{D}_1 = \left\{ \begin{pmatrix} a^{p^{n-r+2}} & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\beta & 1 \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\beta & 1 \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\beta & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\beta & 1 \\ 1 & 1 & 1 & \dots & 1 & a^p & 1 \\ 1 & 1 & 1 & \dots & 1 & 1 & a \end{pmatrix}, \begin{pmatrix} a^{p^{n-r+2}} & 1 & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\kappa \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a^{p^{n-r+2}} & \epsilon_\kappa & 1 & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^p & 1 & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \right\}$$

and obtain a sum of number of cyclic quotients for the first set above in this case as:

$$(p-1)^{r-2}p^{r-3}...p^2p + (p-1)^{r-2}pp^{r-2}p^{r-3}...p^2 + \dots + (p-1)^{r-2}pp^{r-2}p^{r-3}...p$$

And next the above set in this case, is:

$$\mathcal{D}_2 = \left\{ \begin{pmatrix} a & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\gamma \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\alpha \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\alpha \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\alpha \\ 1 & 1 & 1 & \dots & 1 & 1 & a^{p^{n-r+2}} \end{pmatrix}, \begin{pmatrix} a & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\gamma & \epsilon_\kappa \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\gamma & \epsilon_\beta \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\gamma & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\gamma & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^{p^{n-r+2}} & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a & \epsilon_\gamma & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\kappa \\ 1 & a^{p^{n-r+2}} & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \right\}$$

and we obtain a sum of number of cyclic quotients for the above set in this case as:

$$(p^{n-r+2} - p^{n-r+1})^{r-2}p^{n-r+2}p^{r-2}p^{r-3}...p^2p+$$

$$(p-1)^{r-2}p(p^{n-r+2})^{r-2}p^{r-3}...p^2p+\cdots+(p-1)^{r-2}pp^{r-2}p^{r-3}...p^2p^{n-r+2}$$

Continuing with this rule for this case, we finally consider the set:

$$\mathcal{D}_v=\left\{\begin{pmatrix} a^p & \epsilon_\kappa & 1 & \dots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^p & 1 & \dots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a & \dots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^{p^{n-r+2}} & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix}, \begin{pmatrix} a^p & 1 & \epsilon_\gamma & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a & \epsilon_\gamma & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\kappa \\ 1 & 1 & a^{p^{n-r+2}} & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots\ddots & \vdots & \vdots & \vdots & \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix}, \right.$$

$$\dots,$$

$$\left. \begin{pmatrix} a^p & \epsilon_\gamma & 1 & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^{p^{n-r+2}} & 1 & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \right\}$$

and obtain a sum of number of cyclic quotients for this set as:

$$(p-1)^{r-2}pp^{r-2}(p^{n-r+2})^{r-3}...p+(p-1)^{r-2}pp^{r-2}p^{r-3}...(p^{n-r+2})^2$$
$$+\cdots+(p-1)^{r-2}pp^{r-2}p^{r-3}...p^{n-r+2},$$

where

$$|\mathcal{D}_1|+|\mathcal{D}_2|+\cdots+|\mathcal{D}_v|=\ r(r-1)$$

And finally, for the proof of Lemma E to be complete, we consider the next case:

Case n :

In this case, we define

$\mathbb{Z}/p^n\ \cong\ \underbrace{\{\mathbb{Z}^*_{p^{n-r+1}},\mathbb{Z}^*_p,\cdots,\mathbb{Z}^*_p\}}_{(r)-\text{terms}}\ :=\ <a>,\epsilon_\alpha\ \in\ \{a^i\},\quad 1\leq i\leq p^{n-r+1},\quad g\ c\ d(i,p^{n-r+1})=1,$

$\epsilon_\beta\ \in\ \{a^i\},\quad 1\leq i\leq p,\quad g\ c\ d(i,p)=1,$

$\epsilon_\gamma\ \in\ \{a^k\},\quad 0\leq k\leq p^{n-r+1}-1,$

$\epsilon_\kappa\ \in\ \{a^l\},\quad 0\leq l\leq p-1,$

and similarly, applying our fundamental definition together with the counting rule we form the following set of subgroup base representations in $r \ \times r-$ matrices with their respective results:

$$\mathcal{F} = \left\{ \begin{pmatrix} a^{p^{n-r+1}} & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix}, \begin{pmatrix} a^p & \epsilon_\gamma & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^{p^{n-r+1}} & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix}, \right.$$

$$\left. = \begin{pmatrix} a^p & \epsilon_\kappa & \epsilon_\gamma & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^p & \epsilon_\gamma & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a^{p^{n-r+1}} & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} a^p & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & a^p & \dots & \epsilon_\gamma & \epsilon_\kappa & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^{p^{p^{n-r+1}}} & \epsilon_\kappa & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \begin{pmatrix} a^p & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\gamma & \epsilon_\beta \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\gamma & \epsilon_\beta \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\gamma & \epsilon_\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\gamma & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & a^{p^{n-r+1}} & \epsilon_\beta \\ 1 & 1 & 1 & \dots & 1 & 1 & a^p \end{pmatrix} \right.$$

$$\left. \begin{pmatrix} a^p & \epsilon_\kappa & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\alpha \\ 1 & a^p & \epsilon_\kappa & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\alpha \\ 1 & 1 & a^p & \dots & \epsilon_\kappa & \epsilon_\kappa & \epsilon_\alpha \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1 & 1 & 1 & \dots & a^p & \epsilon_\kappa & \epsilon_\alpha \\ 1 & 1 & 1 & \dots & 1 & a^p & \epsilon_\alpha \\ 1 & 1 & 1 & \dots & 1 & 1 & a^{p^{n-r+1}} \end{pmatrix} \right\}$$

and we obtain a sum of number of cyclic quotients for the above set in the last case as:

$$(p-1)^{r-1}p^{r-2}p^{r-3}...p^2p + (p-1)^{r-1}p^{r-2}p^{r-3}...p^2p^{n-r+1}$$
$$+(p-1)^{r-1}p^{r-2}p^{r-3}...(p^{n-r+1})^2p + \dots + (p-1)^{r-1}p^{r-2}(p^{n-r+1})^{r-3}...p^2p+$$
$$(p-1)^{r-1}(p^{n-r+1})^{r-2}p^{r-3}...p^2p + (p^{n-r+1} - p^{n-r})^{r-1}p^{r-2}p^{r-3}...p^2p$$

where $|\mathcal{F}| = \ r$.

Therefore total sums of results obtained in Cases $1, 2, ...,$ to the last yields the formula:

$$p^{(n-1)(r-1)}(\frac{p^r - 1}{p-1}) \quad \square$$

Finally, we give the proof of theorem M.

Theorem M:

Summing from $j = 1$ to n, the number of cyclic quotients up to isomorphism of

$$\underbrace{\mathbb{Z}/p^j \oplus \mathbb{Z}/p^j \oplus \cdots \oplus \mathbb{Z}/p^j}_{r-times}, \quad r > 1, \quad p$$

a prime is then $(\frac{p^r-1}{p-1})(\frac{p^{n(r-1)}-1}{p^{r-1}-1})$.

Proof:

This follows from [1] and Lemma E. $\square$

3 CONCLUSION

This paper solves a very special case of a well-motivated general problem.

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