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The Computation for the Fuzzy Subgroups of the Algebraic Structure $D_{2^4} \times C_{2^4}$

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Abstract: Any finite nilpotent group can be uniquely written as a direct product of p-groups In this paper, an attempt for the computation of $D_{2^4} \times C_{2^4}$ was made. This happens to be the computation of the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order 2^4 with a cyclic group of order sixteen.

Key Words: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral group, inclusion-exclusion principle, maximal subgroups.

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§1. Introduction

The classification of the fuzzy subgroups, most especially the finite p-groups cannot be underestimated. This aspect of pure mathematics has undergone a dynamic developments over the years. For instance, many researchers have treated cases of finite Abelian groups. Since then, the study has been extended to some other important classes of finite Abelian and non-Abelian groups such as the dihedral, quaternion, semidihedral, and hamiltonian groups.

§2. Methodology

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\le r_1<r_2\le t} h(M_{r_1} \cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right). \qquad (2.1)$$

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In [2], (2.1) was used to obtain the explicit formulas for some positive integers n.

Theorem 2.1(Marius) *The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is*

(*i*) $h(\mathbb{Z}_{p^n}) = 2^n$;

(*ii*) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

§3. The number of Fuzzy Subgroups for $\mathbb{Z}_8 \times \mathbb{Z}_8$

Lemma 3.1 *Let G be abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then,* $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$.

Proof By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = h(\mathbb{Z}_2 \times \mathbb{Z}_4)$. And by Theorem 2.1, $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24 \Rightarrow h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$. □

Corollary 3.2 *Following the last lemma,* $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5})$, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6})$, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ *and* $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8})$ *= 1536, 4096, 10496 and 26112 respectively.*

Theorem 3.3 *Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$. Then,* $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$.

Proof Notice that there are three maximal subgroups of G, i.e., one is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}$), while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n}$). We have

$$\begin{aligned}
\frac{1}{2}h(G) &= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\
&= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\
&= h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}})
\end{aligned}$$

Hence,

$$\begin{aligned}
h(G) &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) \\
&= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) \\
&\quad +32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}}) \\
&= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) \\
&\quad +32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}}) \\
&\quad +2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}}).
\end{aligned}$$

For $n - j = 3$

$$\begin{aligned}
&= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})\\
&= 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})\\
&= 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14))\\
&= \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)
\end{aligned}$$

for $n > 2$. ☐

Theorem 3.4 *Suppose that $G = D_{2^3} \times \mathbb{C}_8$. Then, $h(G) = 5376$.*

Proof Notice that

$$\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^3} \times Z_4) + 2h(Z_{2^3} \times Z_2 \times Z_2) - 4h(Z_{2^2} \times Z_2 \times Z_2)\\
&\quad + h(Z_8 \times Z_4) - 6h(Z_8 \times Z_2) - 2h(Z_4 \times Z_4) + 8h(Z_4 \times Z_2) + h(Z_{2^3}) = 2688.
\end{aligned}$$

Therefore $h(G) = 2 \times 2688 = 5376$. ☐

Theorem 3.5 *Let $G = D_{2^5} \times \mathbb{Z}_8$. Then, $h(G) = 111136$.*

Proof Notice that

$$\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^5} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) - 4h(D_{2^4} \times Z_{2^2}) + h(\mathbb{Z}_{2^4} \times Z_{2^3})\\
&\quad - 2h(\mathbb{Z}_{2^4} \times Z_{2^2}) - 2h(\mathbb{Z}_{2^3} \times Z_{2^3}) + 8h(\mathbb{Z}_{2^3} \times Z_{2^2}) + h(\mathbb{Z}_{2^4}) - 4h(\mathbb{Z}_{2^3}) = 55568.
\end{aligned}$$

Therefore, $h(G) = 2 \times 55568 = 111136$. ☐

Theorem 3.6 *Suppose that $G = D_{2^6} \times \mathbb{Z}_8$. Then, $h(G) = 492864$.*

Proof Certainly,

$$\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^6} \times Z_4) + 2h(D_{2^5} \times Z_{2^3}) - 4h(D_{2^5} \times Z_4) + h(\mathbb{Z}_{2^5} \times Z_{2^3})\\
&\quad - 2h(\mathbb{Z}_{2^5} \times Z_{2^2}) - 2h(\mathbb{Z}_{2^4} \times Z_{2^3}) + 8h(\mathbb{Z}_{2^4} \times Z_{2^2}) + h(\mathbb{Z}_{2^5}) - 4h(\mathbb{Z}_{2^4}) = 246432.
\end{aligned}$$

Therefore, $h(G) = 2 \times 246432 = 492864$. ☐

Theorem 3.7 *Let $G = D_{2^n} \times \mathbb{C}_2$. the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by : $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n > 3$.*

§4. The Number of Fuzzy Subgroups for $D_{2^n} \times \mathbb{C}_8$

Proposition 4.1 *Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by*

$$\begin{aligned}
&2^{2(n-1)}(6n+113) + 2^n\left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)\right] \\
&+ \frac{1}{3}(2^{n+2})\Bigg[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 \\
&+ \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]\Bigg].
\end{aligned}$$

Proof Notice that

$$\begin{aligned}
h(D_{2^n} \times C_8) &= 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8) \\
&\quad + 4h(Z_{2^{n-2}} \times C_8) + 2^4 h(Z_{2^{n-3}} \times C_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) \\
&\quad - 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10} h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times C_{2^2}) \\
&\quad + 2^8 h(D_{2^{n-4}} \times C_{2^3}) \\
&= 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) \\
&\quad - 2^{2(n-3)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5} h(\mathbb{Z}_{2^2}) \\
&\quad - 2^{2n-5} h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)} h(D_{2^3} \times \mathbb{Z}_{2^3}) + 3\sum_{i=1}^{n-5} 2^{2ij} h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3})
\end{aligned}$$

as required. □

Proposition 4.2 (see [16]) *Suppose that $G = D_{2^n} \times \mathbb{C}_8$. Then, the number of distinct fuzzy subgroups of G is given by*

$$\begin{aligned}
&2^{2(n-1)}(6n+113) + 2^n\left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j)\right] \\
&+ \frac{1}{3}(2^{n+2})\,[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 \\
&+ \sum_{k=1}^{n-5} 2^k[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58]\Bigg].
\end{aligned}$$

Proof Calculation shows that

$$\begin{aligned}
h(D_{2^n} \times C_8) &= 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times C_8) \\
&\quad + 4h(Z_{2^{n-2}} \times C_8) + 2^4 h(Z_{2^{n-3}} \times C_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times C_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3})
\end{aligned}$$

$$\begin{aligned}
& -4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10}h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times C_{2^2}) \\
& +2^8 h(D_{2^{n-4}} \times C_{2^3}) \\
= \ & 2^n + 2h(D_{2^n} \times C_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) \\
& -2^{2(n-3)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)}h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) - 2^{2n-5}h(\mathbb{Z}_{2^2}) \\
& -2^{2n-5}h(D_{2^3} \times \mathbb{Z}_{2^4}) + 2^{2(n-3)}h(D_{2^3} \times \mathbb{Z}_{2^3}) + 3\sum_{i=1}^{n-5} 2^{2i} h(\mathbb{Z}_{2^{n-i-1}} \times \mathbb{Z}_{2^3})
\end{aligned}$$

as required. □

Theorem 4.3 *Let* $G = D_{2^4} \times \mathbb{C}_{2^4}$. *Then* , $h(G) = 61384$.

Proof There exist seven maximal subgroups. Among them, two isomorphic to $D_{2^4} \times \mathbb{C}_{2^3}$, two isomorphic to $D_{2^3} \times \mathbb{C}_{2^4}$, two isomorphic to $D_{2^4} \times \mathbb{C}_{2^2}$ while the seventh is isomorphic to $\mathbb{Z}_{2^4}$. Hence , we have that

$$\begin{aligned}
\frac{1}{2}h(G) \ = \ & 2h(D_{2^4} \times Z_{2^2}) + 2h(D_{2^4} \times Z_{2^3}) + 2h(D_{2^3} \times Z_{2^4}) \\
& -6h(D_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) - 6h(\mathbb{Z}_{2^4}) \\
& +2h(D_{2^3} \times \mathbb{Z}_{2^3}) + 28h(\mathbb{Z}_{2^3} \times Z_{2^2}) + 2h(Z_{2^4} \times Z_{2^2}) + 2h(\mathbb{Z}_{2^4}) \\
& +h(Z_{2^3} \times \mathbb{Z}_{2^3}) - 35h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + 21h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) - 7h(\mathbb{Z}_{2^3}\mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) \\
= \ & 2[h(D_{2^4} \times Z_{2^2}) + h(D_{2^4} \times Z_{2^3}) + h(D_{2^3} \times Z_{2^4}) - 2h(D_{2^3} \times \mathbb{Z}_{2^3}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) \\
& -h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 4h(D_{2^3} \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_{2^4}) + \frac{1}{2}h(Z_{2^4})].
\end{aligned}$$

And therefore,

$$h(G) = 4[700 + 8416 + 10744 - 10752C1088 + 162 + 704C40] = 4 \times 15346 = 61384. \quad \Box$$

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