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The Number of Chains of Subgroups of the Group $Z_m \times S_n, n \leq 5, m \leq 3$

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Abstract

The study of chains of subgroups in this paper describes the set of all chains of subgroups of G that end in G which is used to solve many computational problems in fuzzy group theory. It is also showed that a fuzzy subgroup is simply a chain of subgroups in the lattice of subgroups.

Keywords: Lattice of Subgroup; Symmetric Group; Subgroup chain; Fuzzy Subgroup.

Mathematics Subject Classification (2020): 20B30, 20B35, 20N25, 20E15.

1 Introduction

Throughout this paper, all groups are assumed to be finite. The lattice of subgroups of a given group G is the lattice $(L(G), \leq)$ where $L(G)$ is the set of all subgroups of G and the partial order $\leq$ is the set inclusion. In this lattice $(L(G), \leq)$, a chain of subgroups of G is a subset of $L(G))$ linearly ordered by set inclusion. A chain of subgroups of G is called rooted (more precisely G-rooted) if it contains G otherwise it is called unrooted.

The study of chains of subgroups in this paper describes the set of all chains of subgroups of G that end in G. Tărnăuceanu and Bentea[5] gave an explicit formula for the number of rooted chains of subgroups of a finite cyclic group by finding its generating function of one variable in the view of classifying the fuzzy subgroups of a finite cyclic group. J.M. Oh in his paper[3] determined the number of subgroups of a finite cyclic group of 4n by finding its generating function of multi variables. Ogiugo and Amit computed the number of chains of subgroups of certain alternating groups using the set of representative of isomorphism classes of subgroups of G(see[9]). The problem of counting chains of subgroups in the lattice of subgroups of a given group G has received attention by researchers with related to classifying fuzzy subgroups of G under a certain type of equivalence relations (see [1],[8], [9] & [10])

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2 Preliminaries

A chain of subgroups of G is a set of subgroups of G totally ordered by set inclusion.The study of chains of subgroups in this paper describes the set of all chains of subgroups of G that end in G.

In this way, suppose that the group G is finite and let $\mu : G \mapsto [0,1]$ be a fuzzy subgroup of G. Put $\mu(G) = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$ and assume that $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then μ determines the following chain of subgroups of G which ends in G:

$$\mu G\alpha_1 \subset \mu G\alpha_2 \subset \cdots \subset \mu G\alpha_m = G$$

Moreover, for any $x \in G$ and $i = \overline{1,r}$, we have

$$\mu(x) = \alpha_i \Leftrightarrow i = max\{j | x \in \mu G\alpha_j\} \Leftrightarrow x \in \mu G_{\alpha_i} \setminus \mu G\alpha_{1-r}$$

A necessary and sufficient condition for two fuzzy subgroups μ, η of G to be equivalent with respect to $\sim$ has been identified in [10] :$\mu \sim \eta$ if and only if μ and η have the same set of level subgroups.In this case, the corresponding equivalence classes of fuzzy subgroups of a group G are closely connected to the chains of subgroups in G. For determining the number of these classes, we calculate the number of all chains of subgroups of G that terminate in G.
Let G be a finite group and $\delta(G)$ be the number of chains of subgroups of G that terminate in G

(1) $G_1 \subset G_2 \subset \cdots \subset G_k = G$ with $G_1 \neq \{e\}$

(2) $\{e\} \subset G_2 \subset \cdots \subset G_k = G$
It is clear that the numbers of chains of types (1) and (2) are equal. So

$$\delta(G) = 2x_k$$

Proposition 2.1
Let $\delta(G)$ be the number of subgroup chains of group G that terminates in G.Then $\delta(G) = \sum_{distictH \in Iso(G)} \delta(H) \times n(H)$
where $Iso(G)$ is the set of representatives of isomorphism classes of subgroups of G, $n(H)$ denotes the size of the isomorphism class with representative H. $\cdots(\#)$

Proof. Let fixes $\delta(H_1) = \delta(H_\alpha) = 1$, for which H_1 is the trivial group of and H_α is the improper subgroup of G For any $H_i \in Iso(G)$ and $i = \overline{1,\alpha}$
Then

$$\begin{aligned}\delta(G) &= n(H_1) * \delta(H_1) + n(H_1) * \delta(H_2) + n(H_3)\delta(H_3) + \cdots + n(H_\alpha)\delta(H_\alpha) \\ &= \sum_{H_i \in Iso(G)} \delta(H_i) \times n(H_i) \\ &= 2 + \sum_{distictH_i \in Iso(G)} \delta(H_i) \times n(H_i)\end{aligned}$$

■

In this work (#) was used to obtain the number of chains of subgroups of G that terminates in G and it was used in to count the number of distinct fuzzy subgroups of finite abelian group([2])

It also follows :

(i) $\delta(Z_p) = 2$ where p is prime

(ii) $\delta(Z_{pq}) = 6$ where p and q are distinct prime

(iii) $\delta(Z_{p^2}) = 4$ where p is any prime

(iv) $\delta(Z_p \times Z_p) = 2p + 4$ where p is any prime

Lemma 2.2 [9]
For groups S_3 A_4 and S_4, we have $\delta(S_3) = 10$, $\delta(A_4) = 24$ and $\delta(S_4) = 232$ respectively.
Lemma 2.3 [7]
Let G be the Dihedral group of order 2p, where p is any prime then , $\delta(G) = 4 + 2p$

3 The number of Chains of Subgroups of $Z_2 \times S_n, n \leq 5$

theorem 3.1 : The number of Chains of Subgroups of $Z_2 \times S_3$ is 68

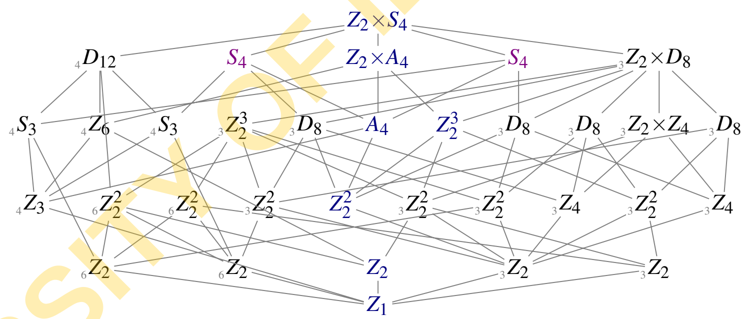
Proof. Let G be $Z_2 \times S_3$ has following set of representatives of isomorphism classes of subgroups with their sizes : $[\{e\},1]$, $[Z_2,7]$, $[Z_2 \times Z_2,3]$, $[Z_3,1]$, $[Z_6,1]$, $[S_3,2]$ and $[Z_2 \times S_3,1]$

$\delta(G) = 1 + \delta(e) + 7 * \delta(Z_2) + \delta(Z_3) + \delta(Z_6) + 2 * \delta(S_3) + 3 * \delta(Z_2 \times Z_2) = 68$
G is isomorphic to Dihedral group of order 12 ■

theorem 3.2 : The number of Chains of Subgroups of $Z_2 \times S_4$ is 3376

Proof. $Z_2 \times S_4$ is a non-abelian group of order 48, it has following set of representatives of isomorphism classes of subgroups with their sizes : $[\{e\},1]$, $[Z_2,19]$, $[Z_3,4]$, $[Z_4,6]$, $[Z_6,4]$
$[Z_2 \times Z_2,25]$, $[Z_4 \times Z_2,3]$, $[Z_2 \times Z_2 \times Z_2,4]$, $[Z_2 \times A_4,1]$, $[Z_2 \times D_8,3]$
$[A_4,1]$, $[D_8,12]$, $[D_{12},4]$, $[S_3,8]$, $[S_4,2]$ and $[Z_2 \times S_4,1]$.

So $\delta(Z_2 \times S_4) = 1 + \delta(H_e) + 19 * \delta(Z_2) + 4 * \delta(Z_3) + 6 * \delta(Z_4) + 4 * \delta(Z_6) + 25 * \delta(Z_2 \times Z_2) + 3 * \delta(Z_4 \times Z_2) + 4 * \delta(Z_2 \times Z_2 \times Z_2) + \delta(Z_2 \times A_4) + 3 * \delta(Z_2 \times D_8) + \delta(A_4) + 12 * \delta(D_8) + 4 * \delta(D_{12}) + 8 * \delta(S_3) + 2 * \delta(S_4) = 3376$
■



Obviously, this problem of counting all chains of subgroups of G depends entirely on the lattice of subgroups of G and not on the group itself.

theorem 3.3 : The number of Chains of Subgroups of group $Z_2 \times S_5$ is 60172

Proof. $Z_2 \times S_5$ is non-abelian of order 240, it has it has following set of representatives of isomorphism classes of subgroups with their sizes : $[\{e\},1]$, $[Z_2,51]$, $[Z_3,10]$, $[Z_4,30]$, $[Z_5,6]$, $[Z_6,30]$, $[Z_{10},6]$ $[Z_2 \times Z_2,105]$, $[Z_4 \times Z_2,15]$, $[(Z_2 \times Z_2 \times Z_2),20]$, $[Z_2 \times A_4,5]$, $[Z_2 \times A_5,1]$, $[(Z_2 \times D_8),15]$
$[(Z_2 \times S_4),5]$, $[(Z_2 \times Z_2 \times S_3),10]$, $[Z_6 \times Z_2,10]$, $[Z_5 \ltimes Z_4,12]$, $[Z_2 \times (Z_5 \ltimes Z_4),6]$
$[A_4,5]$, $[A_5,1]$, $[D_8,60]$, $[D_{10},12]$, $[D_{12},60]$,$[D_{20},6]$, $[S_3,40]$, $[S_4,10]$, $[S_5,2]$ and $[(Z_2 \times S_5),1]$

So $\delta(Z_2 \times S_5) = 1 + \delta(H_e) + 51 * \delta(Z_2) + 10 * \delta(Z_3) + 30 * \delta(Z_4) + 6 * \delta(Z_5) + 30 * \delta(Z_6) + 6 * \delta(Z_{10}) + 105 * \delta(Z_2 \times Z_2) + 15 * \delta(Z_4 \times Z_2) + 20 * \delta(Z_2 \times Z_2 \times Z_2) + 5 * \delta(Z_2 \times A_4) + \delta(Z_2 \times A_5) + 15 * \delta(Z_2 \times D_8) + 5 * \delta(Z_2 \times S_4) + 10 * \delta(Z_2 \times Z_2 \times S_3) + 10 * \delta(Z_6 \times Z_2) + 6 * \delta(Z_4 \ltimes$

$Z_5) + 6 * \delta(Z_2 \times (Z_5 \ltimes Z_4)) + 5 * \delta(A_4) + \delta(A_5) + 60 * \delta(D_8) + 12 * \delta(D_{10}) + 60 * \delta(D_{12}) + 6 * \delta(D_{20}) + 40 * \delta(S_3) + 10 * \delta(S_4) + 2 * \delta(S_5) = 60172$
■

Then, we get the following theorem:

Theorem 3.1. *The number of chains of subgroups $Z_2 \times S_n, n \leq 5$, then*
$$\delta(Z_2 \times S_n) = \begin{cases} 68 & n = 3 \\ 3374 & n = 4 \\ 60172 & n = 5 \end{cases}$$

4 The number of Chains of Subgroups of $Z_3 \times S_n, n \leq 5$

theorem 4.1 : The number of Chains of Subgroups of $Z_3 \times S_3$ is 54

Proof. Let G be $(Z_3 \times S_3)$ has following set of representatives of isomorphism classes of subgroups with their sizes : $[\{e\}, 1]$, $[Z_2, 3]$, $[Z_3 \times Z_3, 1]$, $[Z_3, 4]$, $[Z_6, 3]$, $[S_3, 1]$ and $[(Z_3 \times S_3), 1]$ then $\delta(G) = 1 + \delta(H_e) + 3 * \delta(Z_2) + 4 * \delta(Z_3) + 3 * \delta(Z_6) + \delta(Z_3 \times Z_3) + \delta(S_3) = 54$
■

theorem 4.2 : The number of Chains of Subgroups of $Z_3 \times S_4$ is 1792

Proof. $Z_3 \times S_4$ has following set of representatives of isomorphism classes of subgroups with their sizes : $[\{e\}, 1]$, $[Z_2, 9]$, $[Z_3, 13]$, $[Z_4, 3]$, $[Z_6, 9]$, $[Z_{12}, 3]$ $[Z_2 \times Z_2, 4]$, $[Z_3 \times Z_3, 4]$, $[Z_3 \times S_3, 4]$, $[(Z_6 \times Z_2, 4]$, $[Z_3 \times A_4, 1]$, $[(Z_3 \times D_8), 3]$ $[A_4, 3]$, $[D_8, 3]$, $[S_3, 4]$, $[S_4, 1]$ and $[(Z_3 \times S_4), 1]$

So $\delta(Z_3 \times S_4) = 1 + \delta(Z_1) + 9 * \delta(Z_2) + 13 * \delta(Z_3) + 3 * \delta(Z_4) + 9 * \delta(Z_6) + 3 * \delta(Z_{12}) + 4 * \delta(Z_2 \times Z_2) + 4 * \delta(Z_3 \times Z_3) + 4 * \delta(Z_6 \times Z_2) + 4 * \delta(Z_3 \times Z_3)$
$+ \delta(Z_3 \times A_4) + 3 * \delta(Z_3 \times D_8) + 3 * \delta(A_4) + 3 * \delta(D_8) + 4 * \delta(S_3) + \delta(S_4) = 1792$
■

theorem 4.3 : The number of Chains of Subgroups of $Z_3 \times S_5$ is 33382

Proof. $Z_3 \times S_5$ has following set of representatives of isomorphism classes of subgroups with their sizes : $[\{e\}, 1]$, $[Z_2, 25]$, $[Z_3, 31]$, $[Z_4, 15]$, $[Z_6, 55]$, $[Z_{12}, 15]$, $[Z_2 \times Z_2, 20]$, $[Z_3 \times Z_3, 10]$, $[Z_3 \times S_3, 20]$, $[Z_3 \times (Z_5 \ltimes Z_4), 6]$,$[Z_3 \times S_4, 5]$, $[Z_5 \ltimes Z_4, 6]$, $[Z_6 \times Z_2, 20]$,$[Z_6 \times Z_3, 10]$, $[Z_6 \times S_3, 10]$, $[Z_3 \times A_4, 5]$, $[Z_3 \times A_5, 1]$, $[(Z_3 \times D_8), 15]$, $[(Z_3 \times D_{10}), 6]$ $[A_4, 15]$,$[A_5, 1]$, $[D_8, 15]$, $[D_{10}, 6]$, $[D_{12}, 10]$,$[S_3, 20]$, $[S_4, 5]$, $[S_5, 1]$ and $[(Z_3 \times S_5), 1]$

So $\delta(Z_3 \times A_5) = 1 + \delta(H_e) + 25 * \delta(Z_2) + 31 * \delta(Z_3) + 15 * \delta(Z_4) + 55 * \delta(Z_6) + 15 * \delta(Z_{12}) + 6 * \delta(Z_{15}) + 20 * \delta(Z_2 \times Z_2) + 10 * \delta(Z_3 \times Z_3) + 5 * \delta(Z_3 \times A_4) + 15 * \delta(Z_3 \times D_8) + 6 * \delta(Z_3 \times D_{10}) + 20 * \delta(Z_3 \times S_3) + 5 * \delta(Z_3 \times S_4) + 6 * \delta(Z_3 \times (Z_5 \ltimes Z_4) + 6 * \delta(Z_5 \ltimes Z_4) + 20 * \delta(Z_6 \times Z_2) + 10 * \delta(Z_6 \times Z_3) + 10 * \delta(Z_6 \times S_3)$
$+ 3 * \delta(A_4) + 15 * \delta(D_8) + 6 * \delta(D_{10}) + 10 * \delta(D_{12}) + 20 * \delta(S_3) + 5 * \delta(S_4) + \delta(S_5) = 33382$
■

Then, we get the following theorem:

Theorem 4.1. *The number of chains of subgroups $Z_3 \times S_n, n \leq 5$, then*

$$\delta(Z_3 \times S_n) = \begin{cases} 54 & n = 3 \\ 1792 & n = 4 \\ 33382 & n = 5 \end{cases}$$

5 Conclusion

The study of the number of chains of subgroups in the lattice of subgroups for larger groups are interesting and give rise to potential applications to quantum computing and coding. In this paper, it is observed from the results that the number of chains of subgroups of G does not depend on the order of G but the lattice of subgroups of G

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