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On the p-Groups of the Algebraic Structure of $D_{2^n} \times \mathbb{C}_8$

S. A. Adebisi

(Department of Mathematics, Faculty of Science, University of Lagos, Nigeria)

M. Ogiugo

(Department of Mathematics, School of Science, Yaba College of Technology, Lagos, Nigeria)

M. EniOluwafe

(Department of mathematics,Faculty of Science, university of Ibadan, Nigiria)

E-mail: adesinasunday@yahoo.com

Abstract: In this paper, the explicit formulae is given for the number of distinct fuzzy subgroups of the cartesian product of the dihedral group of order 2^n with a cyclic group of order eight, where $n > 3$.

Key Words: Finite p-groups, nilpotent group, fuzzy subgroups, dihedral Group, inclusion-exclusion principle, maximal subgroups.

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§1. Introduction

This paper is a follow up from[1]. In this work the distinct number of fuzzy subgroups for the Nilpotent p-Group of $D_{2^n} \times \mathbb{Z}_8$ is found.

§2. Methodology

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite p-group G is described. Suppose that $M_1, M_2, \ldots, M_t$ are the maximal subgroups of G, and denote by $h(G)$ the number of chains of subgroups of G which ends in G. By simply applying the technique of computing $h(G)$, using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2\left(\sum_{r=1}^{t} h(M_r) - \sum_{1\leq r_1<r_2\leq t} h(M_{r_1}\cap M_{r_2}) + \cdots + (-1)^{t-1} h\left(\bigcap_{r=1}^{t} M_r\right)\right) \qquad (1.1)$$

In [2], (1.1) was used to obtain the explicit formulas for some positive integers n.

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Theorem A(Marius) *The number of distinct fuzzy subgroups of a finite p-group of order p^n which have a cyclic maximal subgroup is*

(i) $h(\mathbb{Z}_{p^n}) = 2^n$;

(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$.

§3. The Number of Fuzzy Subgroups for $\mathbb{Z}_8 \times \mathbb{Z}_8$

Lemma 3.1 *Let G be Abelian such that $G = \mathbb{Z}_4 \times \mathbb{Z}_4$. Then, $h(G) = 2h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 48$.*

Proof By the use of GAP (Group Algorithms and Programming), G has three maximal subgroups in which each of them is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^2}$. Hence, we have that

$$\begin{aligned} \frac{1}{2}h(G) &= 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) - 3h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) \\ &= h(\mathbb{Z}_2 \times \mathbb{Z}_4). \end{aligned}$$

And by Theorem A, $h(\mathbb{Z}_2 \times \mathbb{Z}_{2^2}) = 24$, which implies that $h(\mathbb{Z}_4 \times \mathbb{Z}_4) = 48$. □

Corrolary 3.2 *Following Lemma* 3.1, $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^5}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^6}), h(\mathbb{Z}_4 \times \mathbb{Z}_{2^7})$ *and* $h(\mathbb{Z}_4 \times \mathbb{Z}_{2^8}) =$ 1536, 4096, 10496 *and* 26112, *respectively.*

Theorem 3.3 *Let $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_8$, then $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$.*

Proof The three maximal subgroups of G have the following properties :

One is isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}$, while two are isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_{2^n}$. We have

$$\begin{aligned} \frac{1}{2}h(G) &= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 3h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\ &= 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) \\ &= h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}). \end{aligned}$$

Hence,

$$\begin{aligned} h(G) &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) - 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}}) \\ &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) - 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) \\ &\quad +32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) - 32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) + 16h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-4}}) \\ &= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-1}}) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-2}}) + 16h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-3}}) \\ &\quad +32h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-4}}) - 64h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-5}}) + 32h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-5}}) \\ &\quad + \cdots - 2^{j+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-j}}) + 2^j h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-j}}) \text{ (for } n-j=3) \end{aligned}$$

$$
\begin{aligned}
&= 4h(\mathbb{Z}_4 \times \mathbb{Z}_{2^n}) + 2^{n-3}h(\mathbb{Z}_8 \times \mathbb{Z}_{2^3}) - 2^{n-1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^3}) + \sum_{k=1}^{n-3}[2^{k+1}h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})\\
&= 2^{n+2}[n^2 + 5n + 3] + \sum_{k=1}^{n-3} h(\mathbb{Z}_4 \times \mathbb{Z}_{2^{n-k}})\\
&= 2^{n+2}((n^2 + 5n + 3) + \frac{1}{6}(n-3)(n^2 + 9n + 14))\\
&= \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)
\end{aligned}
$$

if $n > 2$. This completes the proof. □

Theorem 3.4 *Suppose that $G = D_{2^3} \times \mathbb{C}_8$. Then, $h(G) = 5376$.*

Proof A calculation shows that

$$
\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^3} \times \mathbb{Z}_4) + 2h(\mathbb{Z}_{2^3} \times \mathbb{Z}_2 \times \mathbb{Z}_2) - 4h(\mathbb{Z}_{2^2} \times \mathbb{Z}_2 \times \mathbb{Z}_2)\\
&\quad + h(\mathbb{Z}_8 \times \mathbb{Z}_4) - 6h(\mathbb{Z}_8 \times \mathbb{Z}_2) - 2h(\mathbb{Z}_4 \times \mathbb{Z}_4) + 8h(\mathbb{Z}_4 \times \mathbb{Z}_2)\\
&\quad + h(\mathbb{Z}_{2^3}) = 2688,
\end{aligned}
$$

which implies that $h(G) = 2 \times 2688 = 5376$. This completes the proof. □

Theorem 3.5 *Let $G = D_{2^5} \times \mathbb{Z}_8$. Then, $h(G) = 111136$.*

Proof A calculation shows that

$$
\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^5} \times \mathbb{Z}_{2^2}) + 2h(D_{2^4} \times \mathbb{Z}_{2^3}) - 4h(D_{2^4} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3})\\
&\quad - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^3} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^4})\\
&\quad - 4h(\mathbb{Z}_{2^3}) = 55568,
\end{aligned}
$$

which implies that $h(G) = 2 \times 55568 = 111136$. □

Theorem 3.6 *Suppose that $G = D_{2^6} \times \mathbb{Z}_8$. Then, $h(G) = 492864$.*

Proof A calculation shows that

$$
\begin{aligned}
\frac{1}{2}h(G) &= h(D_{2^6} \times \mathbb{Z}_4) + 2h(D_{2^5} \times \mathbb{Z}_{2^3}) - 4h(D_{2^5} \times \mathbb{Z}_4) + h(\mathbb{Z}_{2^5} \times \mathbb{Z}_{2^3})\\
&\quad - 2h(\mathbb{Z}_{2^5} \times \mathbb{Z}_{2^2}) - 2h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^3}) + 8h(\mathbb{Z}_{2^4} \times \mathbb{Z}_{2^2}) + h(\mathbb{Z}_{2^5}) - 4h(\mathbb{Z}_{2^4}) = 246432,
\end{aligned}
$$

which implies that $h(G) = 2 \times 246432 = 492864$. □

Theorem 3.7 *Let $G = D_{2^n} \times \mathbb{C}_2$, the nilpotent group formed by the cartesian product of the dihedral group of order 2^n and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of G is given by $h(G) = 2^{2n}(2n+1) - 2^{n+1}, n > 3$.*

§4. The Number of Fuzzy Subgroups for $D_{2^n} \times \mathbb{C}_8$

Theorem 4.1 *Suppose that* $G = D_{2^n} \times \mathbb{C}_8$*. Then, the number of distinct fuzzy subgroups of* G *is given by*

$$\begin{aligned} 2^{2(n-1)}(6n+113) \quad &+ \quad 2^n \left[13 - 6n - 2n^2 + 3\sum_{j=1}^{n-3} 2^{(j-1j)}(2n+1-2j) \right] \\ &+ \quad \frac{1}{3}2^{n+2}\left[(n-1)^3 + (n-2)^3 + 24n^2 - 38n - 30 \right. \\ &+ \quad \left. \sum_{k=1}^{n-5} 2^k \left[(n-2-k)^3 + 12(n-2-k)^2 + 17(n-k) - 58\right] \right] \end{aligned}$$

Proof A calculation shows that

$$\begin{aligned} h(D_{2^n} \times \mathbb{C}_8) &= 2h(\mathbb{Z}_{2^{n-1}}) + 2h(D_{2^n} \times Z_4) + 2h(D_{2^{n-1}} \times \mathbb{C}_8) \\ &\quad + 4h(Z_{2^{n-2}} \times \mathbb{C}_8) + 2^4 h(Z_{2^{n-3}} \times \mathbb{C}_8) + 2^6 h(\mathbb{Z}_{2^{n-4}} \times \mathbb{C}_8) - 2^8 h(\mathbb{Z}_{2^{n-5}} \times \mathbb{Z}_{2^3}) \\ &\quad - 4h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) + 2^{10} h(\mathbb{Z}_{2^{n-5}}) \times \mathbb{Z}_{2^2} - 2^9 h(\mathbb{Z}_{2^{n-5}}) - 2^9 h(D_{2^{n-4}} \times \mathbb{C}_{2^2}) \\ &\quad + 2^8 h(D_{2^{n-4}} \times \mathbb{C}_{2^3}) \\ &= 2^n + 2h(D_{2^n} \times \mathbb{C}_4) + 2h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^3}) + 2^2 h(\mathbb{Z}_{2^{n-2}} \times \mathbb{Z}_{2^3}) \\ &\quad - 2^{2(n-3)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^3}) + 2^{2(n-2)} h(\mathbb{Z}_{2^2} \times \mathbb{Z}_{2^2} - 2^2 h(\mathbb{Z}_{2^{n-1}} \times \mathbb{Z}_{2^2}) \\ &\quad - 2^{2n-5} h(\mathbb{Z}_{2^2}) - 2^{2n-5} h(D_{2^3} \times \mathbb{Z}_{2^2}) + 2^{2(n-3)} h(D_{2^3} \times \mathbb{Z}_{2^3}) \\ &\quad + 3\sum_{i=1}^{n-5} 2^{2ij} h(\mathbb{Z}_{2^{n-2-i}} \times \mathbb{Z}_{2^3}) \end{aligned}$$

as required. □

References

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