

## The Generalized Quaternion $p$ -Group of Order $2^n$ : Discovering the Fuzzy Subgroups

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**Abstract.** In this paper, the classification of finite  $p$ -groups is extended to the cartesian product of the generalized quaternion group of order  $2^n$  with a cyclic group of order 2 which also belongs to the class of the famous nilpotent groups .

**Keywords:** Finite  $p$ -groups, nilpotent group, fuzzy subgroups, dihedral group, quaternion group, inclusion-exclusion principle, maximal subgroups

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### 1. Introduction

The following properties for the fuzzy subgroups of  $G$  were known.

1. The level sets of a fuzzy subset of a finite set form a chain.
2.  $\lambda$  is a fuzzy subgroup of  $G$  iff its level sets are subgroups of  $G$
3. The relation  $\sim$  is an equivalence relation on fuzzy subgroups of  $G$ , where

for fuzzy subgroups  $\mu, \nu$  of  $G$ ,  $\mu \sim \nu$  iff  $\forall x, y \in G, (\mu(x) > \mu(y) \text{ iff } \nu(x) > \nu(y))$

Some related algebraic structures are discussed in [7-10].

### 2. Preliminaries

Suppose that  $(G, \cdot, e)$  is a group with identity  $e$ . Let  $S(G)$  denote the collection of all fuzzy subsets of  $G$ . An element  $\lambda \in S(G)$  is said to be a fuzzy subgroup of  $G$  if the following two conditions are sat.

1.  $\lambda(ab) \geq \min\{\lambda(a), \lambda(b)\}, \forall a, b \in G$ ;
2.  $\lambda(a^{-1}) \geq \lambda(a)$  for any  $a \in G$ .

And, since  $(a^{-1})^{-1} = a$ , we have that  $\lambda(a^{-1}) = \lambda(a)$ , for any  $a \in G$ .

Also, by this notation and definition,  $\lambda(e) = \sup \lambda(G)$ . (Marius [3] and [4]).

Now, concerning the subgroups, the set  $FL(G)$  possessing all fuzzy subgroups of  $G$  forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of  $G$ .

In what follows, the method that will be used in counting the chains of fuzzy

S. A. Adebisi and M. EniOluwafe

subgroups of an arbitrary finite  $p$ -group  $G$  is described. Suppose that  $M_1, M_2, \dots, M_t$  are the maximal subgroups of  $G$ . Let  $h(G)$  denote the number of chains of subgroups of  $G$  which ends in  $G$ . The method of computing  $h(G)$  is based on the application of the Inclusion-Exclusion Principle. If  $A$  is the set of chains in  $G$  of type  $C_1 \subset C_2 \subset \dots \subset C_r = G$ , and  $A'$  represents the set of chains of  $A'$  which are contained in  $M_r$ ,  $r = 1, \dots, t$ .

Then we have:

$$\begin{aligned} |A| &= 1 + |A'| = \left| \bigcup_{r=1}^t A_r \right| \\ &= 1 + \sum_{r=1}^t |A_r| - \sum_{1 \leq r_1 < r_2 \leq t} |A_{r_1} \cap A_{r_2}| + \dots + (-1)^{t-1} \left| \bigcap_{r=1}^t A_r \right| \end{aligned}$$

Observe that, for every  $1 \leq w \leq t$  and  $1 \leq r_1 < r_2 < \dots < r_w \leq t$ , the set  $\bigcap_{i=1}^w A_{r_i}$  consists of all chains of  $A'$  which are included in  $\bigcap_{i=1}^w M_{r_i}$ . We have that

$$\begin{aligned} \left| \bigcap_{i=1}^w A_{r_i} \right| &= 2h \left( \bigcap_{i=1}^w M_{r_i} \right)^{-1} \\ \therefore |A| &= 1 + \sum_{r=1}^t (2h(M_r) - 1) - \sum_{1 \leq r_1 < r_2 \leq t} (2h(M_{r_1} \cap M_{r_2}) - 1) \\ &\quad + \dots + (-1)^{t-1} \left( 2h \left( \bigcap_{r=1}^t M_r \right) - 1 \right) \\ &= 2 \left( \sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \dots + (-1)^{t-1} h \left( \bigcap_{r=1}^t M_r \right) \right) + C \end{aligned}$$

And

$$\begin{aligned} C &= 1 + \sum_{r=1}^t (-1) - \sum_{1 \leq r_1 < r_2 \leq t} (-1) + \dots + (-1)^{t-1} (-1) \\ &= (1-1)^t = 0 \end{aligned}$$

we have that:

$$\begin{aligned} h(G) &= 2 \left( \sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) \right. \\ &\quad \left. + \dots + (-1)^{t-1} h \left( \bigcap_{r=1}^t M_r \right) \right) \quad (3) \end{aligned}$$

The Generalized Quaternion  $p$ -Group of Order  $2^n$ : Discovering the Fuzzy Subgroups

(3) was used to obtain the explicit formulas of  $h(D_{2^n})$  for some positive integers  $n$ .

**Theorem 1.** The number of distinct fuzzy subgroups of a finite  $p$ -group of order  $p^n$  which have a cyclic maximal subgroup is:

1.  $h(\mathbf{Z}_{p^n}) = 2^n$
2.  $h(D_{2^n}) = 2^{2n-1}$
3.  $h(\varphi_{2^n}) = 2^{2n-2}$
4.  $h(S_{2^n}) = 3 \cdot 2^{2n-3}$
5.  $h(\mathbf{Z}_p \times \mathbf{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$

Following our paper[1](Also see [2] and [5]) the following equation(#) based on the usual Inclusive-Exclusive technique is applied :

$$h(G) = 2 \left( \sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \dots + (-1)^{t-1} h \left( \bigcap_{r=1}^t M_r \right) \right) (\#)$$

In [4], (1) was used to obtain the explicit formulas of  $h(D_{2^n})$  for some positive integers  $n$ .

**Theorem 2. [3]** The number of distinct fuzzy subgroups of a finite  $p$ -group of order  $p^n$  which have a cyclic maximal subgroup is:

- (i)  $h(\mathbf{Z}_{p^n}) = 2^n$
- (ii)  $h(D_{2^n}) = 2^{2n-1}$
- (iii)  $h(\varphi_{2^n}) = 2^{2n-2}$
- (iv)  $h(S_{2^n}) = 3 \cdot 2^{2n-3}$
- (v)  $h(\mathbf{Z}_p \times \mathbf{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}[2 + (n-1)p]$

### 3. The Cartesian product of the quaternion group of order $2n$ and a cyclic group of order 2

We begin from the simplest form of it, putting  $n = 3$ .

By the application of equation (1), we have:

$$\frac{1}{2}h(G) = 3h(\mathbf{Z}_2 \times \mathbf{Z}_{2^2}) + 4h(Q_{2^3}) - 12h(\mathbf{Z}_{2^2}) - 2h(\mathbf{Z}_2 \times \mathbf{Z}_2) + 8h(\mathbf{Z}_2) = 88$$

$$\therefore h(Q_{2^3} \times C_2) = 2 \times 88 = 176.$$

### 4. Computation of the Fuzzy Subgroups of $Q_{2^4} \times C_2$

Let  $G = Q_{2^4} \times C_2$ . Then, by equation (c)

$$\begin{aligned} \frac{1}{2}h(G) &= h(Z_2 \times Z_{2^3}) + 2h(Q_{2^3} \times C_2) + 4h(Q_{2^4}) - 8h(Q_{2^3}) - 2h(Z_2 \times Z_{2^2}) - \\ &4h(Z_{2^3}) + 8h(Z_{2^2}) = 496 \\ \therefore h(Q_{2^4} \times C_2) &= 2 \times 496 = 992. \end{aligned}$$

By observing the structure of the nilpotent group  $Q_{2^n} \times C_2$ , in general, the group possesses seven maximal subgroups. And so, by using equation (c), putting  $G = Q_{2^n} \times C_2$ , we have

$$\begin{aligned} \frac{1}{2}h(G) &= 2^n + 2^{2^{n-1}} + 2h(Q_{2^{n-1}} \times C_2) \\ \therefore h(Q_{2^n} \times C_2) &= n \cdot 2^{2^n} - 2^{n+1}, \quad n \geq 3. \end{aligned}$$

**Theorem 3.** Let  $G = Q_{2^n} \times C_2$ , the nilpotent group obtained by taking the cartesian product of the generalised quaternion group of order  $2^n$ , and a cyclic group of order 2. Then, the number of distinct fuzzy subgroups of  $G$  is :

$$h(G) = n \cdot 2^{2^n} - 2^{n+1}, \quad \text{for } n > 3$$

**Proof:** One maximal subgroup of this group is isomorphic to  $Z_2 \times Z_{2^{n-1}}$ , two of the maximal subgroups are isomorphic to  $Q_{2^{n-1}} \times C_2$ , while  $2^2$  of the maximal subgroups are isomorphic to  $Q_{2^n}$ . And so, by using equation (3.2.1), putting  $G = Q_{2^n} \times C_2$ , we have

$$\begin{aligned} \frac{1}{2}h(G) &= h(Z_2 \times Z_{2^{n-1}}) + 2h(Q_{2^{n-1}} \times C_2) + 4h(Q_{2^n}) - 6h(Z_{2^{n-1}}) - 3h(Z_2 \times Z_{2^{n-2}}) - \\ &12h(Q_{2^{n-1}}) + 28h(Z_{2^{n-2}}) + 2h(Z_{2^{n-1}}) + h(Z_2 \times Z_{2^{n-2}}) + 4h(Q_{2^{n-1}}) - 20h(Z_{2^{n-2}}) \end{aligned}$$

Putting in the values of the surfacing structures from we have that :

$$h(Q_{2^n} \times C_2) = n \cdot 2^{2^n} - 2^{n+1}, \quad n \geq 3. \quad \text{Applying induction on } n, \text{ by setting } F(n) = h(Q_{2^n} \times C_2), \text{ ascertaining the truth of}$$

$$F(k) = 2^{k+1} + 2^{2^k} + 4h(Q_{2^{k-1}} \times C_2) = k \cdot 2^{2^k} - 2^{k+1}, \quad k \geq 3,$$

we show that  $F(k+1)$  is also true.

$$\text{Hence, } F(k+1) = 2^{k+2} + 2^{2^{k+1}} + 4h(Q_{2^k} \times C_2) = 2^2[(k+1)2^{2^k} - 2^k].$$

**Theorem 4.** ( see [2] and [6]) Suppose that  $G = D_{2^n} \times C_4$ . Then, the number of distinct fuzzy subgroups of  $G$  is given by :

$$2^{2(n-2)}(64n+173) + 3 \sum_{j=1}^{n-3} 2^{(n-1+j)}(2n+1-2j)$$

## 6. Conclusion

Finally, the product of the generalised quaternion  $p$ -group of order  $2^n$  and a cyclic

The Generalized Quaternion  $p$ -Group of Order  $2^n$ : Discovering the Fuzzy Subgroups

group of order 2 have been successfully classified and the number of distinct fuzzy subgroups were directly computed using comprehensive analysis and the application of GAP(Group Algorithms and Programming, Version 4.8.7; <https://www.gap-system.org>)

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