Nigerian Journal of Mathematics and Applications
Volume **30**, (2020), 73 − 78. *Printed by Unilorin press*

http://www.njmaman.com

On the Nilpotent Fuzzy Subgroups of the Abelian Type: $\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}, n \geq 5$

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Abstract

The extension of the finite nilpotent groups is now being diversified. As such, results up to two dimensions are now obtainable. In this paper, the fuzzy subgroups of the nilpotent abelian structure given by: $\mathbb{Z}_{32} \times \mathbb{Z}_{2^n}$, the cartesian product of two abelian subgroups of orders 2^n and 32 respectively for every integer $n > 5$ have been carefully studied and the explicit formulae for its number distinctly given.

1. Introduction

Before this time, several methods, techniques and approaches were used for the classification of which some are obtainable in [5], [6] and [8]. For example, the natural equivalence relation was introduced in [10]. In this work, solution to the problems defined is given using the technique which is known as the normal Inclusion-Exclusion Principle. The aftermath of this method gives rise to some forms of common relations. This paves way to reasonable results and the

Received: 17/09/2020, Accepted: 03/11/2020, Revised: 11/11/2020. * Corresponding author.
2015 *Mathematics Subject Classification.* 20D15, 20E28, 20F18, 20N25, 20K27; Secondary 08A05, 08A72, 20E34, 20K01 & 20K35.
Keywords and phrases. Finite p-Groups, Abelian Group, Fuzzy subgroups, isomorphic, Inclusion-Exclusion Principle, Maximal subgroups, Nilpotent group & Explicit formulae
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problems are completely solved easily by using some simple computations as required. In the process of our computation, the use of GAP(Group Algorithm and Programming) was actually applied.

1.1. **Basic Definitions and Terms.** Let $(H, \cdot, e)$ be a group which has an identity denoted by e. Collection of all fuzzy subsets of H, and denote this by $P(H)$.

Definition 1.1. [7]: An element $\tau \in P(H)$ would be known as a fuzzy subgroup of H provided that each of the conditions given below hold and are strictly maintained: $\tau(mn) \geq min\{\tau(m), \tau(n)\}$ for all $m, n \in H$ and that $\tau(m^{-1} \geq \tau(m))$ whenever $m \in H$. Now since $(m^{-1})^{-1} = m$, it implies that $\tau(m^{-1}) = \tau(m)$, as $m \in H$. Following similar definitions, $\tau(e) = \sup \tau(H)$.

We shall denote the subgroups of H which are maximal in their own right in H by $M_1, M_2, \cdots, M_t$. Now let $h(H)$ be the number of chains of subgroups of H whose ends lie in H.

Theorem 1.2. [7]*: The lattice under the usual ordering of fuzzy set inclusion is hereby represented by the set denoted by* $G(H)$.

This contains every subgroup of H which are fuzzified and referred to as the fuzzy subgroup lattice of H. In the same vein define as follows:

Definition 1.3. [7]: The level subset: $\tau H_\delta = \{m \in H | \tau(m) \geq \delta\}$ for each $\delta \in [0, 1]$.

By this definition, the fuzzy subgroups of a finite p-group H are thus, characterized, according to the subsets. Note that, τ is a fuzzy subgroup of H provided the level subsets are contained in H.
In practice however, certain l relations on $P(H)$ can also be used naturally, when dealing with the concept of classifying the subgroups which are fuzzified for every finite q-group H (see [9] and [10]). One of them is defined by: $\tau \sim \gamma$ iff $(\tau(m) > \gamma(n) \Longleftrightarrow \tau(m) > \gamma(n), \ \forall\, m, n \in H)$. Also, two fuzzy subgroups τ, γ of H are said to be distinct if $\tau \nsim \gamma$.
As a result of this development, let H be a finite p-group. Now, suppose that $\tau : H \longrightarrow [0, 1]$ is fuzzy for H. Set $\tau(H) = \{\beta_1, \beta_2, \ldots, \beta_k\}$ with the assumption that $\beta_1 > \beta_2 > \cdots > \beta_k$. Then, H is determined by τ following the chain given below:

$$\tau H_{\beta_1} \subset \lambda G_{\beta_2} \subset \cdots \subset \tau H_{\beta_k} = H \qquad (a)$$

Also, we have that:

$$\tau H(a) = \beta_t \Longleftrightarrow t = \max\{r / a \in \tau H_{\beta_r}\} \Longleftrightarrow a \in \tau H_{\beta_t} \backslash \tau H_{\beta_{t-1}},$$

for any $a \in H$ and $t = 1, \ldots, k$. Here, we set $\tau H_{\beta_0} = \phi$, conventionally as required.

2. The Techniques

Here, we clarify the technique commonly applied for the process of getting the chains of fuzzy subgroups of an arbitrary finite p-group H. Let $V_1, V_2, \cdots, V_k$ be the subgroups which are maximal for H. Denote by $h(H)$, the number of chains of subgroups of H whose ends lie in H. What we do is to apply the method which computes $h(H)$, by simply using the technique of the Inclusion-Exclusion Principle. It follows from here that:

$$(1)\quad h(H) = 2\left(\sum_{b=1}^{k} h(V_b) - \sum_{1\leq b_1<b_2\leq k} h(V_{b_1}\cap V_{b_2}) + \cdots + (-1)^{k-1} h\left(\bigcap_{b=1}^{k} V_b\right)\right)$$

In [8], (1) was used to obtain the explicit formulas for some positive integers n.

Theorem 2.1. [10]*: The number of every distinct fuzzy subgroups of a p-group which is finite and possesses the order that is equal to* p^n *and having maximal subgroups whose one is cyclic is given by: (i)* $h(\mathbb{Z}_{p^n}) = 2^n$
(ii) $h(\mathbb{Z}_p \times \mathbb{Z}_{p^{n-1}}) = h(M_{p^n}) = 2^{n-1}(2 + (n-1)p)$

3. The Fuzzy Subgroups for $\mathbb{Z}_{16} \times \mathbb{Z}_{16}$

The abelian structure possesses three similar maximal subgroups $A \sim \mathbb{Z}_8 \times \mathbb{Z}_{16}$

Lemma 3.1. [2]*: Let G be abelian such that* $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}, n \leq m$ *then, there exists* $p+1$ *maximal subgroups in G.*

Lemma 3.2. [1]*: Let* $Q = \mathbb{Z}_8 \times \mathbb{Z}_{2^n}$*. Then, for* $n > 2, h(Q)$
$= \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$

Proposition 3.3. [3]*: Let* $G = \mathbb{Z}_{16} \times \mathbb{Z}_{16}, h(G) = 6400$

Proof: Using Lemma 3.2 and by carefully applying equation (1), we have that: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_8 \times \mathbb{Z}_16) - 3h(\mathbb{Z}_8 \times \mathbb{Z}_{16}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{16})$. Thus $h(G) = 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4}) = 6400$ □.

Proposition 3.4. *: Suppose that* $G = \mathbb{Z}_{2^5} \times \mathbb{Z}_{16}$*. Then,* $h(G) = 27136$

Proof: By (1), $\frac{1}{2}h(G) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{16}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4}), h(G) = 27136$ □.

Lemma 3.5. *: If* $G = \mathbb{Z}_{2^6} \times \mathbb{Z}_{16}, then h(G) = 95488$

Proof: Using (1), $\frac{1}{2}h(G) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) \therefore h(G) = 95488$ □.

Theorem 3.6. *Suppose that* $G = \mathbb{Z}_{2^n} \times \mathbb{Z}_{16}$.
Then $h(G) = \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$

$$+ \quad \frac{1}{3}(2^{n+1})\sum_{k=1}^{n-5}[(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24]$$

Proof: The three maximal subgroups, one is isomorphic to $\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}$, while two are isomorphic to $\mathbb{Z}_8 \times \mathbb{Z}_{2^n}$ for $n > 4$.
We have: $h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})]$
$= \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$

$$+ \quad \frac{1}{3}(2^{n+1})\sum_{k=1}^{n-5}[(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24]$$

□.

Proposition 3.7. *(i)*[2]*: Suppose that* $G = Z_4 \times Z_{2^n}, n \geq 2$. *Then,* $h(G) = 2^n[n^2 + 5n2]$

Proposition 3.8. *(ii)*[3]*: Let* $G = \mathbb{Z}_8 \times \mathbb{Z}_{2^n}$. *Then* $h(G) = \frac{1}{3}(2^{n+1})(n^3 + 12n^2 + 17n - 24)$

Proposition 3.9. *(iii)*[4]*: Suppose that* $G = Z_{16} \times \mathbb{Z}_{2^n}$. *Then* $h(G) = \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$

$$+\frac{1}{3}(2^{n+1})\sum_{k=1}^{n-5}[(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24].$$

Lemma 3.10. *Let* $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^5}$, *then* $h(G) = 13124$

Proof: $\frac{1}{2}h(G) = 3h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^4})3h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^4}) + h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^4}) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) = 6562$, by (iii) above. $\therefore h(G) = 2 \times 6562 = 13124$ □.

Lemma 3.11. [3]*: Let* $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^6}$, *then,* $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6})$
$- h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5})$
$+ h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5})$.
$\therefore h(G) = 2[h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5})]$

Lemma 3.12. [3]*: Let* $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^7}$, *then,* $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7})$
$- 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6})$
$\therefore h(G) = 2[h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^6}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^6})]$

Lemma 3.13. [4]*: Let* $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^8}$, *then,* $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^7}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^8})$
$- h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^8}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7}) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^7})$
$+ h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^8}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^7})$.

Lemma 3.14. *Let* $G = \mathbb{Z}_{32} \times \mathbb{Z}_{2^n}$ *Then*

$$h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})] + \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}).$$

Proof: For any integer $n > 5$, we have three maximal subgroups which are isomorphic to $\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}, \mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$ and $\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$.
Then, $\frac{1}{2}h(G) = h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})$.
$\therefore h(G) = 2h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-1}}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) - 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})$
$= 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) - 4h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-2}}) + 4h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-2}})$
$= 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + 2h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) + 4h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-2}}) - 8h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-3}})$
$+ 8h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^{n-3}})$.
$hence h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})]$

$$+ \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}) 2^{n-5} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + 2^{n-5} h(\mathbb{Z}_{32} \times \mathbb{Z}_{2^5})$$

$= 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})]$

$$+ \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}) - 13124 + 13124$$

$$= 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}) + h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}})] + \sum_{k=1}^{n-6} 2^{k+1} h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1-k}}).$$

as required □.

Acknowledgement: The authors are grateful to University of Lagos, Yaba College of Technology and University of Ibadan for the supports they received during the compilation of this work.

Competing interests: The manuscript was read and approved by all the authors. They therefore declare that there is no conflicts of interest.

Funding: The Authors received no financial support for the research, authorship, and/or publication of this article.

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