

# FUZZY SUBGROUPS FOR (THE CARTESIAN PRODUCT OF) THE ABELIAN STRUCTURE : $\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}, n > 3$

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## Abstract

As part of the extension of the finite nilpotent groups to the direct product of  $p$ -groups, we give in this paper, the explicit formulae for the number of distinct fuzzy subgroups of the abelian structure given by:  $\mathbb{Z}_{16} \times \mathbb{Z}_{2^n}, n > 3$ , the Cartesian product of two abelian groups of orders  $2^n$  and 16 respectively for every integer  $n > 3$ .

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## 1. Introduction

Different approaches were used for the classification of which some are obtainable in [2] and [5]. For example, the natural equivalence relation was introduced in [7]. In this work, an essential role in solving counting problems is played by adopting the Inclusion-Exclusion Principle. The process leads to some recurrence relations from which the solutions are finally computed with ease. In the process of our computation, the use of GAP (Group Algorithm and Programming) was actually applied.

## 2. Basic Definitions and Terms

Suppose that  $(G, \cdot, e)$  is a group with identity  $e$ . Let  $S(G)$  denote the collection of all fuzzy subsets of  $G$ . An element  $\lambda \in S(G)$  is said to be a fuzzy subgroup of  $G$  if the following two conditions are satisfied

(i)  $\lambda(ab) \geq \min\{\lambda(a), \lambda(b)\}, \forall a, b \in G$ ; (ii)  $\lambda(a^{-1}) \geq \lambda(a)$  for any  $a \in G$ .

And, since  $(a^{-1})^{-1} = a$ , we have that  $\lambda(a^{-1}) = \lambda(a)$ , for any  $a \in G$ .

Also, by this notation and definition,  $\lambda(e) = \sup\lambda(G)$ . [Marius [4]]. Define by  $M_1, M_2, \dots, M_t$ , the maximal subgroups of  $G$ , and denote by  $h(G)$  the number of chains of subgroups of  $G$  which ends in  $G$ .

**Theorem [Marius [4]]:** The set  $FL(G)$  possessing all fuzzy subgroups of  $G$  forms a lattice under the usual ordering of fuzzy set inclusion. This is called the fuzzy subgroup lattice of  $G$ .

We define the level subset:  $\lambda G_\beta = \{a \in G / \lambda(a) \geq \beta\}$  for each  $\beta \in [0, 1]$  The fuzzy subgroups of a finite  $p$ -group  $G$  are thus, characterized, based on these subsets. In the sequel,  $\lambda$  is a fuzzy

subgroup of  $G$  if and only if its level subsets are subgroups in  $G$ . This theorem gives a link between  $FL(G)$  and  $L(G)$ , the classical subgroup lattice of  $G$ .

Moreover, some natural relations on  $S(G)$  can also be used in the process of classifying the fuzzy subgroups of a finite  $q$ -group  $G$  (see [6] and [7]). One of them is defined by:  $\lambda \sim \gamma$  iff  $(\lambda(a) > \lambda(b) \iff v(a) > v(b), \forall a, b \in G)$ . Also, two fuzzy subgroups  $\lambda, \gamma$  of  $G$  are said to be distinct if  $\lambda \not\sim \gamma$ .

As a result of this development, let  $G$  be a finite  $p$ -group and suppose that  $\lambda : G \rightarrow [0, 1]$  is a fuzzy subgroup of  $G$ . Put  $\lambda(G) = \{\beta_1, \beta_2, \dots, \beta_k\}$  with the assumption that  $\beta_1 < \beta_2 < \dots < \beta_k$ . Then, ends in  $G$  is determined by  $\lambda$ .

$$\lambda G_{\beta_1} \subset \lambda G_{\beta_2} \subset \dots \subset \lambda G_{\beta_k} = G \quad (a)$$

Also, we have that:

$$\lambda(a) = \beta_t \iff t = \max\{r/a \in \lambda G_{\beta_r}\} \iff a \in \lambda G_{\beta_t} \setminus \lambda G_{\beta_{t-1}}, \text{ for any } a \in G \text{ and } t = 1, \dots, k, \text{ where by convention, set } \lambda G_{\beta_0} = \emptyset.$$

### 3. The Techniques

The method that will be used in counting the chains of fuzzy subgroups of an arbitrary finite  $p$ -group  $G$  is described. Suppose that  $M_1, M_2, \dots, M_t$  are the maximal subgroups of  $G$ , and denote by  $h(G)$  the number of chains of subgroups of  $G$  which ends in  $G$ . By simply applying the technique of computing  $h(G)$ , using the application of the Inclusion-Exclusion Principle, we have that:

$$h(G) = 2 \left( \sum_{r=1}^t h(M_r) - \sum_{1 \leq r_1 < r_2 \leq t} h(M_{r_1} \cap M_{r_2}) + \dots + (-1)^{t-1} h \left( \bigcap_{r=1}^t M_r \right) \right) \quad (1)$$

In [5], (#) was used to obtain the explicit formulas for some positive integers  $n$ .

**Theorem (\*) [Marius][7]:** The number of distinct fuzzy subgroups of a finite  $p$ -group of order  $p^n$  which have a cyclic maximal subgroup is:

- (i)  $h(\mathbb{Z}_p^n) = 2^n$
- (ii)  $h(\mathbb{Z}_p \times \mathbb{Z}_p^{n-1}) = h(M_p n) = 2^{n-1}(2 + (n - 1)p)$

### 4. The Fuzzy Subgroups for

$$\mathbb{Z}_{16} \times \mathbb{Z}_{16}$$

The abelian structure possesses three similar maximal subgroups which are isomorphic to  $\mathbb{Z}_8 \times \mathbb{Z}_{16}$

**Lemma 1:** Let  $G$  be abelian such that  $G = \mathbb{Z}_2 \times \mathbb{Z}_2^n$   $m, n \leq m$  then, there exists  $p + 1$  maximal subgroups in  $G$ .

**Lemma 2 [1] :** Let  $Q = \mathbb{Z}_8 \times \mathbb{Z}_2^n$ . Then, for  $n > 2$ ,  $h(Q) = \frac{1}{3}(2^{+1})(n^3 + 12n^2 + 17n - 24)_n$

**Proposition 3:** Let  $G = \mathbb{Z}_{16} \times \mathbb{Z}_{16}$ ,  $h(G) = 6400$

**Proof :** Using lemma 2 and by carefully applying equation (1), we have that :

$$\frac{1}{2}h(G) = 3h(\mathbb{Z}_8 \times \mathbb{Z}_{16}) - 3h(\mathbb{Z}_8 \times \mathbb{Z}_{16}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{16}).$$

Thus  $h(G) = 2h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4}) = 6400$  (by lemma 2 )

**Proposition 4:** Suppose that  $G = \mathbb{Z}_{16} \times \mathbb{Z}_{2^5}$ . Then,  $h(G) = 27136$

**Proof :** By (1),  $\frac{1}{2}h(G) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{16}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^4})$ . Hence,  $h(G) = 27136$ .

**Lemma 5 :**  $G = \mathbb{Z}_{16} \times \mathbb{Z}_{p^{2^6}}$  ,  $h(G) = 95488$

**Proof :** Using (1),  $\frac{1}{2}h(G) = h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^5}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^6}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^5})$  . $\therefore h(G) = 95488$

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**Theorem :** Suppose that  $G = \mathbb{Z}_{16} \times \mathbb{Z}_{2^n}$ .

Then  $h(G) = \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$

$$+ \frac{1}{3}(2^{n+1}) \sum_{k=1}^{n-5} [(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24]$$

**Proof:** The three maximal subgroups, one is isomorphic to  $\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}$ , while two are isomorphic to  $\mathbb{Z}_8 \times \mathbb{Z}_{2^n}$  for  $n > 4$ .

We have :  $h(G) = 2[h(\mathbb{Z}_{16} \times \mathbb{Z}_{2^{n-1}}) + h(\mathbb{Z}_8 \times \mathbb{Z}_{2^n}) - h(\mathbb{Z}_8 \times \mathbb{Z}_{2^{n-1}})]$

$$= \frac{1}{3}(2^{n+2})(n^3 + 12n^2 + 17n - 24) + 2^n(200)$$

$$+ \frac{1}{3}(2^{n+1}) \sum_{k=1}^{n-5} [(n-k)^3 + 12(n-k)^2 + 17(n-k) - 24]$$

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